Solutions to

H. W. 6.

1 Ans.

$$1 \det \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$+5 \det \begin{pmatrix} 2 & 2 & 1 \\ 5 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$1 (-1) \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$+5 (-1) \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= (-1) [2 - 1] - 5 [2 - 1]$$

$$= -1 - 5 = -6$$

2 Ans.

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} \alpha + 2\beta + 3\gamma = 0 \\ \left.\begin{array}{l} 2\alpha + \beta + \gamma = 0 \\ \left.\begin{array}{l} \alpha + 5\beta = 0 \\ \alpha = 0 \end{array}\right\} \Rightarrow \beta = 0 \end{array}\right\} \Rightarrow \gamma = 0 \end{array}$$

Hence $\alpha = \beta = \gamma = 0$ is the only solution.

The vectors are linearly independent.

3 The vectors have to be linearly independent.

$$\alpha\begin{pmatrix}1\\2\\3\\4\end{pmatrix}+\beta\begin{pmatrix}1\\2\\3\\0\end{pmatrix}+\gamma\begin{pmatrix}1\\2\\0\\0\end{pmatrix}+\delta\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Rightarrow \alpha+\beta+\gamma+\delta=0$$

$$2\alpha+2\beta+2\gamma=0 \quad \Rightarrow \gamma=0$$

$$3\alpha+3\beta=0 \quad \Rightarrow \beta=0$$

$$4\alpha=0 \Rightarrow \alpha=0$$

$$\delta=0.$$

$\therefore \alpha=\beta=\gamma=\delta=0$

is the only solution.

vectors are linearly independent.

In $\mathbb{R}^4$, any 4 l.i. vectors form a basis. To show that we ~~n~~ need to show that any vector.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

can be written as a linear combination of the 4 vectors, ie. $\exists\, \alpha, \beta, \gamma, \delta$:

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Caution: These $\alpha, \beta, \gamma, \delta$ are not the same as that on page 3.

$$\alpha + \beta + \gamma + \delta = x$$
$$2\alpha + 2\beta + 2\gamma = y.$$
$$3\alpha + 3\beta = z.$$
$$4\alpha = \omega.$$

I $4\alpha = \omega \Rightarrow \alpha = \frac{1}{4}\omega.$

II $3\alpha + 3\beta = z \Rightarrow 3\beta = z - 3\alpha = z - \frac{1}{4}\omega.$

$$\Rightarrow \beta = \frac{1}{3}z - \frac{1}{12}\omega.$$

III $2\alpha + 2\beta + 2\gamma = y$

$$\Rightarrow 2\gamma = y - 2\alpha - 2\beta.$$
$$= y - \frac{1}{2}\omega - \frac{2}{3}z + \frac{1}{6}\omega.$$
$$= y - \frac{2}{3}z - \frac{1}{3}\omega.$$
$$\Rightarrow \gamma = \frac{1}{2}y - \frac{1}{3}z - \frac{1}{6}\omega.$$

IV $\quad \alpha + \beta + \gamma + \delta = x$

$$\Rightarrow \delta = x - \alpha - \beta - \gamma$$

$$= x - \frac{1}{4}\omega - \cancel{\frac{1}{3}z} + \frac{1}{12}\omega - \frac{1}{2}y + \cancel{\frac{1}{3}z} + \frac{1}{6}\omega$$

$$= x - \cancel{\frac{1}{4}\omega} + \cancel{\frac{1}{12}\omega} + \cancel{\frac{1}{6}\omega} - \frac{1}{2}y$$

$$= x - \frac{1}{2}y$$

It follows that

$$\begin{pmatrix} x \\ y \\ z \\ \omega \end{pmatrix} = \frac{1}{4}\omega \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \left(\frac{1}{3}z - \frac{1}{12}\omega\right) \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} +$$

$$\left(\frac{1}{2}y - \frac{1}{3}z - \frac{1}{6}\omega\right) \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \left(x - \frac{1}{2}y\right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We have shown that any arbitrary vector $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ can be written as a linear combination of the four vectors in this problem.

Hence the vectors form a basis.

4 Ans:

Let us write

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \end{pmatrix}, v_4 = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 4 \end{pmatrix}$$

The vectors $v_1, v_2, v_3, v_4$ are linearly dependent. (Please check that).

In fact we can see that

$$v_4 = v_1 + v_2 + v_3.$$

Hence every vector in the span of $v_1, v_2, v_3, v_4$ must be also in the span of vectors $v_1, v_2, v_3$.

$$\text{span}\{v_1, v_2, v_3, v_4\} = \text{span}\{v_1, v_2, v_3\}.$$
$$= V.$$

The vectors $v_1, v_2, v_3$ are also linearly dependent (Please check this).

In fact

$$v_3 = v_1 - v_2.$$

Hence every vector in the span of $\{v_1, v_2, v_3\}$ must also be in the span of vectors $\{v_1, v_2\}$.

we conclude that

$$\text{span}\{v_1, v_2, v_3, v_4\} = \text{span}\{v_1, v_2, v_3\}.$$

$$= \text{span}\{v_1, v_2\} = V.$$

The vectors $v_1, v_2$ are linearly independent. To see this write

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} \alpha + \beta &= 0. \\ 2\alpha + 2\beta &= 0. \\ 3\alpha + 3\beta &= 0. \\ 4\alpha = 0 &\Rightarrow \alpha = 0 \end{aligned}\right\} \Rightarrow \beta = 0.$$

$\therefore \alpha = \beta = 0$ is the only solution.

V is spanned by 2 independent vectors $v_1$ & $v_2$. Hence a basis of V is given by $\{v_1, v_2\}$.

dimension of V is 2.

the # of elements in any basis.

(5) To calculate co-ordinates, we write

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$$\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} \alpha + \beta + \gamma &= 3. \\ \beta + \gamma &= 7 \\ \gamma &= 9 \end{aligned}\right\} \left.\beta = 7 - 9 = -2 \right\} \begin{aligned} \alpha &= 3 + 2 - 9 \\ &= 5 - 9. \\ &= -4. \end{aligned}$$

$$\therefore \begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix} = (-4) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-2) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 9 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore (-4, -2, 9)$ are the three co-ordinates of $\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}$ w.r.t. the chosen basis.

6 Ans:

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : A \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 0 \\ 3 & 6 & 3 & 9 & 0 \\ 5 & 8 & 3 & 13 & 0 \end{pmatrix} \leftarrow \text{Augmented matrix.}$$

$$\longleftrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & -2 & 0 \end{pmatrix}$$

$$\longleftrightarrow \begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \Rightarrow \begin{aligned} x - z + w &= 0 \\ y + z + w &= 0 \end{aligned}$$

$$\Rightarrow \begin{aligned} x &= z - w. \\ y &= -z - w. \end{aligned}$$

If $z = s$, $w = t$ we have

$$x = s - t$$
$$y = -s - t$$

$$W = \left\{ \begin{pmatrix} s-t \\ -s-t \\ s \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

$$\begin{pmatrix} s-t \\ -s-t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$W = \text{span} \left\{ \underset{a_1}{\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}}, \underset{a_2}{\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}} \right\}.$$

The vectors $\theta_1, \theta_2$ are linearly independent and span the null space. Hence they form a basis.

dimension of the null space $= 2$.

7. The rows of A are

$(1 \quad 2 \quad 1 \quad 3)$,

$(3 \quad 6 \quad 3 \quad 9)$,

$(5 \quad 8 \quad 3 \quad 13)$

$$R = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 3 \\ 9 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 3 \\ 13 \end{pmatrix} \right\}.$$

These vectors are written column wise.

Check independence:

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 6 \\ 3 \\ 9 \end{pmatrix} + \gamma \begin{pmatrix} 5 \\ 8 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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Aug matrix.

$$\begin{pmatrix} 1 & 3 & 5 & 0 \\ 2 & 6 & 8 & 0 \\ 1 & 3 & 3 & 0 \\ 3 & 9 & 13 & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \alpha + 3\beta + 5\gamma &= 0 \\ \gamma &= 0 \end{aligned} \Rightarrow \begin{aligned} \alpha &= -3\beta. \\ \gamma &= 0. \end{aligned}$$

<u>conclusion</u>

$$\begin{pmatrix}1\\2\\1\\3\end{pmatrix}, \begin{pmatrix}3\\6\\3\\9\end{pmatrix}, \begin{pmatrix}5\\8\\3\\13\end{pmatrix}$$ are linearly dependent.

since $\alpha = -3\beta$ we have.
$\gamma = 0$

$$-3\beta\begin{pmatrix}1\\2\\1\\3\end{pmatrix} + \beta\begin{pmatrix}3\\6\\3\\9\end{pmatrix} + 0\begin{pmatrix}5\\8\\3\\13\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

choose $\beta = 1$ and we get

$$\begin{pmatrix}3\\6\\3\\9\end{pmatrix} = 3\begin{pmatrix}1\\2\\1\\3\end{pmatrix}$$

It follows that

$$R = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 3 \\ 13 \end{pmatrix} \right\}.$$

The second vector can be dropped because it is 3 times the first vector.

The above two vectors are linearly independent.

$$\boxed{\text{Hence } \dim R = 2.}$$

$$\mathcal{C} = \text{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 13 \end{pmatrix} \right\}$$

To check independence let us write

$$\alpha \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + \delta \begin{pmatrix} 3 \\ 9 \\ 13 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Aug. matrix

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 0 \\ 3 & 6 & 3 & 9 & 0 \\ 5 & 8 & 3 & 13 & 0 \end{pmatrix} \longleftrightarrow$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$\leftarrow$ already done on page 12.

Hence

$\alpha + -\gamma + \delta = 0 \Rightarrow \alpha = \gamma - \delta$

$\beta + \gamma + \delta = 0 \Rightarrow \beta = -\gamma - \delta.$

It follows that

$\gamma = 1, \delta = 0, \alpha = 1, \beta = -1.$

is one solution which would imply.

$$\begin{pmatrix}1\\3\\5\end{pmatrix} - \begin{pmatrix}2\\6\\8\end{pmatrix} + \begin{pmatrix}1\\3\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1\\3\\3\end{pmatrix} = \begin{pmatrix}2\\6\\8\end{pmatrix} - \begin{pmatrix}1\\3\\5\end{pmatrix}.$$

It also follows that.

$\gamma = 0, \delta = 1, \alpha = -1, \beta = -1.$

is another solution.

It follows that

$$-\begin{pmatrix}1\\3\\5\end{pmatrix}-\begin{pmatrix}2\\6\\8\end{pmatrix}+\begin{pmatrix}3\\9\\13\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}3\\9\\13\end{pmatrix}=\begin{pmatrix}1\\3\\5\end{pmatrix}+\begin{pmatrix}2\\6\\8\end{pmatrix}$$

It follows finally that

$$\mathcal{C}=\operatorname{span}\left\{\begin{pmatrix}1\\3\\5\end{pmatrix},\begin{pmatrix}2\\6\\8\end{pmatrix},\begin{pmatrix}1\\3\\3\end{pmatrix},\begin{pmatrix}3\\9\\13\end{pmatrix}\right\}$$

$$=\operatorname{span}\left\{\begin{pmatrix}1\\3\\5\end{pmatrix},\begin{pmatrix}2\\6\\8\end{pmatrix}\right\}$$

IIIrd & IVth vectors are l.d. on the 1st two vectors.

$\therefore \begin{pmatrix}1\\3\\5\end{pmatrix}$ & $\begin{pmatrix}2\\6\\8\end{pmatrix}$ are l.i.

it follows that they form a basis of $\mathcal{C}$.

dimension of $\mathcal{C} = 2$.

Thus we verify that

$$\boxed{\dim \mathcal{R} = \dim \mathcal{C} = 2}$$

$\dim \mathcal{N} = 2$ (problem 6).

$\therefore \dim \mathcal{N} + \dim \mathcal{R} = 2 + 2 = 4$

$= \#$ columns of the matrix $A$. $\blacksquare$