HW 5 - SOLUTIONS

(1)

Clearly V is closed under addition and multiplication defined.

Now we have to verify if the Axioms on page 119 hold.

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$.

$x_1, x_2, y_1, y_2, z_1, z_2, \alpha, \beta$ are all real.

A1: $x \oplus y = (x_1 y_1 - x_2 y_2, \; x_1 y_2 + x_2 y_1) = y \oplus x$

A2: $(x \oplus y) \oplus z = (x_1 y_1 - x_2 y_2, \; x_1 y_2 + x_2 y_1) \oplus (z_1, z_2)$

$= (x_1 y_1 z_1 - x_2 y_2 z_1 - x_1 y_2 z_2 - x_2 y_1 z_2, \; x_1 y_1 z_2 - x_2 y_2 z_2 + x_1 y_2 z_1 + x_2 y_1 z_1)$

$x \oplus (y \oplus z) = (x_1, x_2) \oplus (y_1 z_1 - y_2 z_2, \; y_1 z_2 + y_2 z_1)$

$= (x_1 y_1 z_1 - x_1 y_2 z_2 - x_2 y_1 z_2 - x_2 y_2 z_1, \; x_1 y_1 z_2 + x_1 y_2 z_1 + x_2 y_1 z_1 - x_2 y_2 z_2)$

$\therefore$ Clearly $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

A3: Consider $(1, 0)$:

Then, $(1, 0) \oplus (x_1, x_2) = (1 \cdot x_1 - 0 x_2, \; 1 \cdot x_2 + 0 \cdot x_1)$

$= (x_1, x_2)$

and $(x_1, x_2) \oplus (1, 0) = (x_1 \cdot 1 - x_2 \cdot 0, \; x_1 \cdot 0 + x_2 \cdot 1)$

$= (x_1, x_2)$

$\therefore$ $(1, 0)$ is the "zero" element!

A4 : Given any element $x = (x_1, x_2)$ find (a_1, a_2) such that $(x_1, x_2) \oplus (a_1, a_2) = (1, 0)$

Then, $(x_1 a_1 - x_2 a_2 , x_1 a_2 + x_2 a_1) = (1, 0)$

i.e. $x_1 a_1 - x_2 a_2 = 1$ — ①

$x_1 a_2 + x_2 a_1 = 0$ — ②

If $x_1 = 0$ & $x_2 = 0$, then, we cannot solve these two equations for a_1, a_2.

$\therefore$ We cannot find a "$-x$" for $x = (0, 0)$.

You can stop here but details of the other axioms are given below.

For $x_1 \neq 0$ & $x_2 = 0$:

$$a_1 = \frac{1}{x_1} \quad \text{and} \quad a_2 = 0$$

For $x_1 = 0$ & $x_2 \neq 0$

$$a_1 = 0 \quad \text{and} \quad a_2 = -\frac{1}{x_2}$$

For $x_1 \neq 0$ and $x_2 \neq 0$,

$$a_1 = -\frac{x_1 a_2}{x_2} \quad \text{from ②}$$

now from ①, $-\frac{x_1^2 a_2}{x_2} - x_2 a_2 = 1$

$$-x_1^2 a_2 - x_2^2 a_2 = x_2$$

$$\therefore a_2 = \frac{-x_2}{x_1^2 + x_2^2}$$

and $a_1 = \frac{x_1}{x_1^2 + x_2^2}$

Note that this general formula works even when one of x_1 <u>or</u> x_2 is non zero. But will <u>not</u> work if both are zero.

i.e. if $x_1 = x_2 = 0$, then, there is no "$-x$" such the $x + (-x) = (1, 0)$ (Remember $(1,0)$ is "zero" here

Hence given V is <u>not a vector</u> space.

But for other cases,

$$-x = \left(\frac{x_1}{x_1^2 + x_2^2}, \frac{-x_2}{x_1^2 + x_2^2} \right)$$

when not both x_1 & x_2 are zero.

You can verify that,

$$x \oplus (-x) = (x_1, x_2) \oplus \left(\frac{x_1}{x_1^2 + x^2}, \frac{-x_2}{x_1^2 + x_2^2} \right)$$

$$= \left(\frac{x_1^2}{x_1^2 + x_2^2} - \frac{-x_2^2}{x_1^2 + x_2^2}, \frac{x_1 x_2}{x_1^2 + x_2^2} - \frac{x_2 x_1}{x_1^2 + x_2^2} \right)$$

$$= \left(\frac{x_1^2 + x_2^2}{x_1^2 + x_2^2}, 0 \right)$$

$$= (1, 0)$$

As: $\alpha \odot (x \oplus y) = \alpha \odot ((x_1, x_2) \oplus (y_1, y_2))$

$$= \alpha \odot ((x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1))$$

$$= (\alpha + x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1)$$

But $(\alpha \odot x) \oplus (\alpha \odot y) = (\alpha + x_1, x_2) \oplus (\alpha + y_1, y_2)$

$$= ((\alpha + x_1)(\alpha + y_1) - x_2 y_2, x_2(\alpha + y_1) + y_2(\alpha + x_1))$$

$$= (\alpha^2 + x_1 y_1 + \alpha y_1 + \alpha x_1 - x_2 y_2, \alpha x_2 + \alpha y_2 + x_2 y_1 + y_2 x_1)$$

$\therefore$ Clearly $\alpha \odot (x \oplus y) \neq (\alpha \odot x) \oplus (\alpha \odot y)$

A6: $(\alpha+\beta)\odot x = (\alpha+\beta+x_1, x_2)$

$$\alpha\odot x + \beta\odot x = (\alpha+x_1, x_2) \oplus (\beta+x_1, x_2)$$
$$= ((\alpha+x_1)(\beta+x_1) - x_2^2, (\alpha+x_1)x_2 + (\beta+x_1)x_2)$$

Clearly this one fails too.

i.e. $(\alpha+\beta)\odot x \neq \alpha\odot x + \beta\odot x$.

A7: $(\alpha\beta)\odot x = (\alpha\beta + x_1, x_2)$

$$\alpha\odot(\beta\odot x) = \alpha\odot(\beta+x_1, x_2)$$
$$= (\alpha+\beta+x_1, x_2)$$

$\therefore$ $(\alpha\beta)\odot x \neq \alpha\odot(\beta\odot x)$ too.

$\therefore$ This one fails

A8: The "one" for this is actually 0

because, $0\odot(x_1, x_2) = (0+x_1, x_2) = (x_1, x_2)$

for any $x = (x_1, x_2)$

Axioms A1, A2, A3, A8 works for any $x, y, z \in V$.

But A4, A5, A6, A7 fails.

A7 could be the easiest to check to see that V is not a vector space.

(2)

(i) $W = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 0\} \subset \mathbb{R}^4$.

For any $a \in \mathbb{R}$, $(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) \in W$

$a(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) = (a\hat{x}_1, a\hat{x}_2, a\hat{x}_3, a\hat{x}_4)$

Then $a\hat{x}_1 + a\hat{x}_2 + a\hat{x}_3 + a\hat{x}_4 = a(\underbrace{\hat{x}_1 + \hat{x}_2 + \hat{x}_3 + \hat{x}_4}_{=0}) = 0$

$\therefore a(x_1, x_2, x_3, x_4) \in W$.

For any $(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) \in W$ & $(\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4) \in W$

$(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) + (\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4) = (\hat{x}_1 + \hat{y}_1, \hat{x}_2 + \hat{y}_2, \hat{x}_3 + \hat{y}_3, \hat{x}_4 + \hat{y}_4)$

Then,

$(\hat{x}_1 + \hat{y}_1) + (\hat{x}_2 + \hat{y}_2) + (\hat{x}_3 + \hat{y}_3) + (\hat{x}_4 + \hat{y}_4)$

$= (\underbrace{\hat{x}_1 + \hat{x}_2 + \hat{x}_3 + \hat{x}_4}_{=0}) + (\underbrace{\hat{y}_1 + \hat{y}_2 + \hat{y}_3 + \hat{y}_4}_{=0})$

$= 0$

$\therefore (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) + (\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4) \in W$.

$\therefore$ W is a subspace of V.

(ii) $W = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1\}$.

For any $a \in \mathbb{R}$, $(\hat{x}_1, \hat{x}_2, \hat{x}_3) \in W$

$$a(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (a\hat{x}_1, a\hat{x}_2, a\hat{x}_3)$$

Then check

$$a\hat{x}_1 + a\hat{x}_2 + a\hat{x}_3 = a\underbrace{(\hat{x}_1 + \hat{x}_2 + \hat{x}_3)}_{=1} = a \neq 1$$

If W was a subspace we need $a(x_1, x_2, x_3) \in W$ (i.e. $ax_1 + ax_2 + ax_3 = 1$) This does not happen for <u>every</u> a (this happens for $a = 1$ only, so it is not good enough)

$\therefore$ W is <u>not a</u> subspace of V.

(iii) $W = \{(x_1, x_2) : x_1 = 3x_2\}$

For any $a \in \mathbb{R}$, $(\hat{x}_1, \hat{x}_2) \in W$ (i.e. $\hat{x}_1 = 3\hat{x}_2$)

$$a(\hat{x}_1, \hat{x}_2) = (a\hat{x}_1, a\hat{x}_2)$$

Then check, $a\hat{x}_1 = a(3\hat{x}_2)$ (because $\hat{x}_1 = 3\hat{x}_2$)

$$(a\hat{x}_1) = 3(a\hat{x}_2)$$

$\therefore$ $a(\hat{x}_1, \hat{x}_2) \in W$

For any $(\hat{x}_1, \hat{x}_2)$, $(\hat{y}_1, \hat{y}_2) \in W$,

$$(\hat{x}_1, \hat{x}_2) + (\hat{y}_1, \hat{y}_2) = (\hat{x}_1 + \hat{y}_1, \hat{x}_2 + \hat{y}_2)$$

Then check $\hat{x}_1 + \hat{y}_1 = (3\hat{x}_2) + (3\hat{y}_2)$ because, $\hat{x}_1 = 3\hat{x}_2$ & $\hat{y}_1 = 3\hat{y}_2$

$$\hat{x}_1 + \hat{y}_1 = 3(\hat{x}_2 + \hat{y}_2)$$

$\therefore$ $(\hat{x}_1, \hat{x}_2) + (\hat{y}_1, \hat{y}_2) \in W$ $\therefore$ W is a subspace of V.

(iv) $W = \{(x_1, x_2, x_3) : x_1 = x_2 = x_3\}$

For any $(\hat{x}_1, \hat{x}_2, \hat{x}_3) \in W$ and $a \in \mathbb{R}$, (i.e. $\hat{x}_1 = \hat{x}_2 = \hat{x}_3$)

$$a(x_1, x_2, x_3) = (ax_1, ax_2, ax_3)$$

$$a\hat{x}_1 = a\hat{x}_2 \quad (\text{because } \hat{x}_1 = \hat{x}_2)$$

and $a\hat{x}_1 = a\hat{x}_3$ (because $\hat{x}_1 = \hat{x}_3$)

i.e. $a\hat{x}_1 = a\hat{x}_2 = a\hat{x}_3$

$\therefore a(x_1, x_2, x_3) \in W$.

For any $(\hat{x}_1, \hat{x}_2, \hat{x}_3) \in W$, $(\hat{y}_1, \hat{y}_2, \hat{y}_3) \in W$,

$$(\hat{x}_1, \hat{x}_2, \hat{x}_3) + (\hat{y}_1, \hat{y}_2, \hat{y}_3) = (\hat{x}_1 + \hat{y}_1, \hat{x}_2 + \hat{y}_2, \hat{x}_3 + \hat{y}_3)$$

Then check,

$\hat{x}_1 + \hat{y}_1 = \hat{x}_2 + \hat{y}_2$ (because $\hat{x}_1 = \hat{x}_2$ & $\hat{y}_1 = \hat{y}_2$).

and $\hat{x}_1 + \hat{y}_1 = \hat{x}_3 + \hat{y}_3$ (because $\hat{x}_1 = \hat{x}_3$ & $\hat{y}_1 = \hat{y}_3$).

$\therefore (\hat{x}_1 + \hat{y}_1) = (\hat{x}_2 + \hat{y}_2) = (\hat{x}_3 + \hat{y}_3)$

$\therefore (\hat{x}_1, \hat{x}_2, \hat{x}_3) + (\hat{y}_1, \hat{y}_2, \hat{y}_3) \in W$

$\therefore$ W <u>is a</u> subspace of V.

(v) Upper triangular 2×2 matrices are, of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$.

Then for any $\alpha \in \mathbb{R}$, and, $\begin{pmatrix} \hat{a} & \hat{b} \\ 0 & \hat{c} \end{pmatrix} \in W$

$\alpha \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ 0 & \alpha c \end{pmatrix}$ is still upper triangular. So, it is in W

For any another $\begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{c} \end{pmatrix} \in W$,

$\begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{c} \end{pmatrix} + \begin{pmatrix} \hat{a} & \hat{b} \\ 0 & \hat{c} \end{pmatrix} = \begin{pmatrix} \tilde{a}+\hat{a} & \tilde{b}+\hat{b} \\ 0 & \tilde{c}+\hat{c} \end{pmatrix}$ is also upper triangular, so it is also in W.

$\therefore$ Upper triangular matrices are a subspace of all 2×2 matrices.

(vi) The process is the same. So W is a sub space

(vii) All polynomials in W are quadratic polynomials which has the same coefficient in all terms. i.e. any "element" in W can be written as $a(x^2+x+1)$. So for any $\alpha \in \mathbb{R}$,

$\alpha(a(x^2+x+1)) = \alpha a(x^2+x+1)$ is of the same form

and $(a_1(x^2+x+1)) + (a_2(x^2+x+1)) = (a_1+a_2)(x^2+x+1)$ is also of the same form. $\therefore$ They are both in W. So it is a subspace

③ (i) (a) $\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\6\end{pmatrix}$ in $\mathbb{R}^2$.

Set $a\begin{pmatrix}2\\3\end{pmatrix} + b\begin{pmatrix}4\\6\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

Then, $2a + 4b = 0$
$3a + 6b = 0$

The augmented matrix: $\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 3 & 6 & 0 \end{array}\right) \xrightarrow{R_1/2} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 6 & 0 \end{array}\right) \xrightarrow{R_2/3} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right) \xrightarrow{R_2 - R_1 \to R_2} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

i.e. $a + 2b = 0$.

Set $b = t$, then, $a = -2t$

∴ we can find two non zero numbers a & b.

∴ These two vectors are linearly <u>dependendant</u>

(You may also could have made this observation by inspection, because $\begin{pmatrix}4\\6\end{pmatrix} = 2 \cdot \begin{pmatrix}2\\3\end{pmatrix}$).

(b) We would need at least two independant vectors to span $\mathbb{R}^2$, so these two vetors will not span $\mathbb{R}^2$.

(c) Since these two are not linearly independant, Only one vector is linearly independant

∴ $\dim\{\operatorname{span}(\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\6\end{pmatrix})\} = 1$

(d) A "natural" basis would be just $\begin{pmatrix}2\\3\end{pmatrix}$.

(ii) (a) Writing $\alpha\begin{pmatrix}2\\3\\1\end{pmatrix}+\beta\begin{pmatrix}4\\6\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, we get:

$\begin{pmatrix}2&4\\3&6\\1&1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ and the augmented matrix:

$$\left(\begin{array}{cc|c}2&4&0\\3&6&0\\1&1&0\end{array}\right)\xrightarrow{R_2/2}\left(\begin{array}{cc|c}1&2&0\\3&6&0\\1&1&0\end{array}\right)\xrightarrow{R_2-3R_1\to R_2}\left(\begin{array}{cc|c}1&2&0\\0&0&0\\1&1&0\end{array}\right)$$

$$\xrightarrow{R_1-R_3\to R_1}\left(\begin{array}{cc|c}0&1&0\\0&0&0\\1&1&0\end{array}\right)\xrightarrow{R_2\leftrightarrow R_3}\left(\begin{array}{cc|c}0&1&0\\1&1&0\\0&0&0\end{array}\right)\xrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{cc|c}1&1&0\\0&1&0\\0&0&0\end{array}\right)$$

$$\xrightarrow{R_1-R_2\to R_1}\left(\begin{array}{cc|c}1&0&0\\0&1&0\\0&0&0\end{array}\right)$$

$\therefore$ $\alpha=0,\ \beta=0$.

$\therefore$ These two vectors are linearly independent.

(b) These two vectors cannot span $\mathbb{R}^3$. Because it is necessary to have 3 vectors to span $\mathbb{R}^3$.

(c) $\dim\left\{\operatorname{span}\left(\begin{pmatrix}2\\3\\1\end{pmatrix},\begin{pmatrix}4\\6\\1\end{pmatrix}\right)\right\}=2$.

(d) These two should span a 2-dimensional sub space in $\mathbb{R}^3$

The easiest basis to compute would be

$\begin{pmatrix}4\\6\\1\end{pmatrix}-\begin{pmatrix}2\\3\\1\end{pmatrix}=\begin{pmatrix}2\\3\\0\end{pmatrix}$ and, $\begin{pmatrix}2\\3\\1\end{pmatrix}-\begin{pmatrix}2\\3\\0\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$

So, another basis would be $\left\{\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\3\\0\end{pmatrix}\right\}$.

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(iii) (a) Writing $\alpha\begin{pmatrix}0\\1\\1\end{pmatrix}+\beta\begin{pmatrix}1\\0\\1\end{pmatrix}+\gamma\begin{pmatrix}1\\1\\0\end{pmatrix}+\delta\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ we get,

$$\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},$$ and the augmented matrix

$$\left(\begin{array}{cccc|c}0&1&1&1&0\\1&0&1&1&0\\1&1&0&1&0\end{array}\right)\xrightarrow{R_3-R_2\to R_3}\left(\begin{array}{cccc|c}0&1&1&1&0\\1&0&1&1&0\\0&1&-1&0&0\end{array}\right)$$

$$\xrightarrow{R_3-R_1\to R_3}\left(\begin{array}{cccc|c}0&1&1&1&0\\1&0&1&1&0\\0&0&-2&-1&0\end{array}\right)\xrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{cccc|c}1&0&1&1&0\\0&1&1&1&0\\0&0&-2&-1&0\end{array}\right)$$

$$\xrightarrow{R_3/-2}\left(\begin{array}{cccc|c}1&0&1&1&0\\0&1&1&1&0\\0&0&1&1/2&0\end{array}\right)\xrightarrow{R_2-R_3\to R_2}\left(\begin{array}{cccc|c}1&0&1&1&0\\0&1&0&1/2&0\\0&0&1&1/2&0\end{array}\right)$$

$$\xrightarrow{R_1-R_3\to R_1}\left(\begin{array}{cccc|c}1&0&0&1/2&0\\0&1&0&1/2&0\\0&0&1&1/2&0\end{array}\right)$$

This is the most we can reduce.

$\therefore$ $\alpha+\frac{1}{2}\delta=0$, $\beta+\frac{1}{2}\delta=0$, $\gamma+\frac{1}{2}\delta=0$.

$\therefore$ Set $\delta=t$, then,

$\alpha=-\frac{3}{2}t$, $\beta=-\frac{3}{2}t$, $\gamma=\frac{1}{2}t$.

$\therefore$ We can find all non zero $\alpha,\beta,\gamma,\delta$.

$\therefore$ Not linearly independant.

You may also could have made the observati that,

$$\begin{pmatrix}0\\1\\1\end{pmatrix}+\begin{pmatrix}1\\0\\1\end{pmatrix}+\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\2\\2\end{pmatrix}=2\begin{pmatrix}1\\1\\1\end{pmatrix}$$

(b) + (c) :

Since $\dim\left(\text{span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}\right)=3$,

and $\dim(\mathbb{R}^3)=3$,

we can span $\mathbb{R}^3$ with these 4 vectors.

(even though one vector is redundant)

d) $\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}$, the natural basis of $\mathbb{R}^3$.

(iv) Writing $\alpha\begin{pmatrix}2\\1\\1\end{pmatrix}+\beta\begin{pmatrix}1\\2\\1\end{pmatrix}+\gamma\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

we get $\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$.

The augmented matrix:

$$\left(\begin{array}{ccc|c}2&1&1&0\\1&2&1&0\\1&1&2&0\end{array}\right)\xrightarrow{R_1-R_2\to R_1}\left(\begin{array}{ccc|c}1&-1&0&0\\1&2&1&0\\1&1&2&0\end{array}\right)\xrightarrow{R_2-R_3\to R_2}\left(\begin{array}{ccc|c}1&-1&0&0\\0&1&-1&0\\1&1&2&0\end{array}\right)$$

$$\xrightarrow{R_3-R_1\to R_3}\left(\begin{array}{ccc|c}1&-1&0&0\\0&1&-1&0\\0&2&2&0\end{array}\right)\xrightarrow{R_3/2\to R_3}\left(\begin{array}{ccc|c}1&-1&0&0\\0&1&-1&0\\0&1&1&0\end{array}\right)\xrightarrow{R_2+R_3\to R_2}\left(\begin{array}{ccc|c}1&-1&0&0\\0&2&0&0\\0&1&1&0\end{array}\right)$$

$$\xrightarrow{R_2/2\to R_2}\left(\begin{array}{ccc|c}1&-1&0&0\\0&1&0&0\\0&1&1&0\end{array}\right)\xrightarrow{R_3-R_2\to R_3}\left(\begin{array}{ccc|c}1&-1&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)\xrightarrow{R_1+R_2\to R_1}\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)$$

$\therefore\ \alpha=\beta=\gamma=0$

$\therefore$ These three vectors are linearly independant.

(b) + (c)

$$\dim\left(\operatorname{span}\left\{\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right\}\right) = 3$$

and $\dim(\mathbb{R}^3) = 3$.

$\therefore$ The given 3 vectors spans $\mathbb{R}^3$.

(d) a natural basis is

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(v) Set $a\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+b\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+c\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+d\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$.

Then, $\begin{pmatrix}0 & a\\ a & 0\end{pmatrix}+\begin{pmatrix}b & 0\\ 0 & b\end{pmatrix}+\begin{pmatrix}0 & c\\ -c & 0\end{pmatrix}+\begin{pmatrix}-d & 0\\ 0 & d\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

i.e. $\begin{pmatrix}b-d & a+c\\ a-c & b+d\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

we get,

$$\left.\begin{aligned} b-d &= 0\\ a+c &= 0\\ a-c &= 0\\ b+d &= 0\end{aligned}\right\} \Rightarrow \begin{pmatrix}0 & 1 & 0 & -1\\ 1 & 0 & 1 & 0\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\\ 0\end{pmatrix}$$

The augmented matrix:

$$\left(\begin{array}{cccc|c}0 & 1 & 0 & -1 & 0\\ 1 & 0 & 1 & 0 & 0\\ 1 & 0 & -1 & 0 & 0\\ 0 & 1 & 0 & 1 & 0\end{array}\right) \xrightarrow{R_2+R_3\to R_3} \left(\begin{array}{cccc|c}0 & 1 & 0 & -1 & 0\\ 1 & 0 & 1 & 0 & 0\\ 2 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0\end{array}\right) \xrightarrow{R_1+R_4\to R_1} \left(\begin{array}{cccc|c}0 & 2 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0\\ 2 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0\end{array}\right)$$

$$\xrightarrow{R_1/2\to R_1} \left(\begin{array}{cccc|c}0 & 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0\\ 2 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0\end{array}\right) \xrightarrow{R_3/2\to R_3} \left(\begin{array}{cccc|c}0 & 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0\end{array}\right)$$

$$\xrightarrow{R_2-R_3\to R_2} \left(\begin{array}{cccc|c}0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0\end{array}\right) \xrightarrow{R_4-R_1\to R_4} \left(\begin{array}{cccc|c}0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{array}\right)$$

From this we get $a=b=c=d=0$

$\therefore$ These 4 matrices are linearly independent.

(b) & (c)

These 4 can span the space of 2 x 2 matrices, and have a dimension 4.

(d) A natural basis will be,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$