Home Work 4

① Let A be an $n \times n$ matrix. Is it possible for $A^2 + I = 0$ in the case where n is odd? How about if n is even.

② Consider the 3×3 Vandermonde matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$$

show that

$$\det V = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$$

Hint: Make use of row operation $\underline{\text{III}}$.

What can you say about the 4×4 Vandermonde matrix.

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}$$

(3) Let A be a skew-symmetric matrix

(a) ie $A^T = -A$.

Show that if A is an $n \times n$ skew symmetric matrix and n is odd then A must be singular.

(b) Let A be a 3×3 singular matrix. It must be of the form

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}$$

Calculate $\det A$ using the co-factor formula and show that $\det A = 0$.

(4) If A is a $n \times n$ matrix then

$$A \cdot \operatorname{adj} A = (\det A) \cdot I$$

Start with a 3×3 matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

(a) write down $\operatorname{adj} A$.

(b) write down the product $A \cdot \operatorname{adj} A$ and show that the product equals

$$\begin{pmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{pmatrix}$$

(c) If A is a singular matrix, what can you say about $A \cdot \operatorname{adj} A$.

(5) Let R be a $n \times n$ matrix such that

$$R R^T = I$$

(a) Show that $\det R$ is either 1 or -1.

(b) For $n=2$ show that R must be of the form.

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

for some parameter θ.

(c) For $n=3$ show that the matrix

$$R = \begin{pmatrix} .3481 & .6313 & .6930 \\ .9332 & -.3038 & -.1920 \\ .0893 & .7135 & -.6949 \end{pmatrix}$$

satisfies $RR^T = I$ and $\det R = 1$.