Linear Algebra
Homework 2

§1.4

1 (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ - Elementary matrix
Interchange row 1 & 2 (Type 1)

(b) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ - Not an elementary matrix
(But this is the product of 2 elementary marices of type 2)

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}$ - Elementary matrix
Add 5 times row 1 to row 3 (Type 3)

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ - Elementary matrix
multiplies row 2 by 5 (Type 2)

3 (a) $A = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}$; $B = \begin{pmatrix} -4 & 2 \\ 5 & 3 \end{pmatrix} \Rightarrow E = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$; $B = EA$

(b) $A = \begin{pmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{pmatrix}$; $B = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{pmatrix} \Rightarrow E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$; $B = EA$

(c) $A = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{pmatrix}$; $B = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{bmatrix} \Rightarrow E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$; $B = EA$

4 (a) $A = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 1 & 4 \\ 4 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \Rightarrow E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; $B = AE$

(b) $A = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$; $B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix} \Rightarrow E = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$; $B = AE$

(c) $A = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 4 & 2 \\ 6 & 1 & -2 \end{bmatrix}$; $B = \begin{bmatrix} 2 & -2 & 3 \\ -1 & 4 & 2 \\ 3 & 1 & -4 \end{bmatrix} \Rightarrow E = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $B = AE$

⑤ $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{bmatrix}$

a) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ then $EA = B$.

b) $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ then $FB = C$

c) Yes because, $C = FB = F(EA) = (FE)A$.

⑥ $A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix}$

a) Note that

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix} \xrightarrow{R_2 - 3R_1 \to R_2} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \xrightarrow{R_3 - 2R_1 \to R_3} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_3 \to R_3} U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$\therefore$ set $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$; $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

(b) $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$; $E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$; This is lower triangular

Clearly $LU = A$. (This is called L-U decomposition)

7. $A = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}$

(a) Note that

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{2R_1 \to R_1} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 + R_2 \to R_1} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{3R_1 + R_2 \to R_2} \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} = A$$

$$\therefore \quad A = \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{E_3} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{E_2} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{E_1}$$

(b)

$$\therefore A^{-1} = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1/2 \\ -3 & 1 \end{bmatrix}$$

11 $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

A^{-1} calculation:

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1 + R_2 \rightarrow R_2} \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ -1 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{3R_2 + R_1 \rightarrow R_1} \left[\begin{array}{cc|cc} 0 & 1 & -5 & 3 \\ -1 & 0 & -2 & 1 \end{array}\right] \xrightarrow{-R_2 \rightarrow R_2} \left[\begin{array}{cc|cc} 0 & 1 & -5 & 3 \\ 1 & 0 & 2 & -1 \end{array}\right]$$

$$\xrightarrow[R_2 \rightarrow R_1]{R_1 \rightarrow R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -5 & 3 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

a) $AX = B \Rightarrow X = A^{-1}B = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\therefore X = \begin{bmatrix} -1 & 0 \\ 4 & 2 \end{bmatrix}$$

(b) $YA = B \Rightarrow Y = BA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

$$\therefore Y = \begin{bmatrix} -8 & 5 \\ -14 & 9 \end{bmatrix}$$

21 (a) Let $D_1 = \begin{bmatrix} a_1 & & & \\ & a_2 & & 0 \\ & 0 & \ddots & \\ & & & a_n \end{bmatrix}$ & $D_2 = \begin{bmatrix} b_1 & & & \\ & b_2 & & 0 \\ & 0 & \ddots & \\ & & & b_n \end{bmatrix}$

$$D_1 D_2 = \begin{bmatrix} a_1 b_1 & & & \\ & a_2 b_2 & & 0 \\ & 0 & \ddots & \\ & & & a_n b_n \end{bmatrix} = D_2 D_1$$

(b) $B = a_0 I + a_1 A + \dots + a_k A^k$

$AB = a_0 A + a_1 A^2 + \dots + a_k A^{k+1} = BA.$

§2.1

(1) $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$

(a) $\det(M_{21}) = \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = -8$

$\det(M_{22}) = \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = -2$

$\det(M_{23}) = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5$

(b) $A_{21} = 8$; $A_{22} = -2$, $A_{23} = -5$

(c) $\det(A) = a_{21} \cdot A_{21} + a_{22} \cdot A_{22} + a_{23} \cdot A_{23}$

$= 8 + 4 - 15$

$= -3$

* Remember $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

2 (a) $\det\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix} = 12-10 = 2 \neq 0 \Rightarrow$ Non Singular

(b) $\det\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} = 12-12 = 0 \Rightarrow$ Singular

(c) $\det\begin{pmatrix} 3 & -6 \\ 2 & 4 \end{pmatrix} = 12+12 = 24 \neq 0 \Rightarrow$ non singular

3 (c) $\begin{vmatrix} 3 & 1 & 2 \\ 2 & 4 & 5 \\ 2 & 4 & 5 \end{vmatrix} = 0$ (2^{nd} & 3^{rd} row are the same)

(g) $\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix}$

$= 2\begin{vmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 & 0 \\ 1 & 6 & 2 \\ 1 & 1 & -2 \end{vmatrix}$

$= 2(6) - 1(-1)(-2-2)$

$= 8$

4 (a) $\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} = 2$

(b) $\begin{vmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 3 & -2 \end{vmatrix} = -4$ (product of the diagonal terms)

(c) $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 0$ (because row1 + row3 = 2 × row2)

(d) $\begin{vmatrix} 4 & 0 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 4 \\ 1 & 0 & 2 & 3 \end{vmatrix} = 0$ (one column is entirely zero).

6 $\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix}$

$$= (3-\lambda)(2-\lambda) - 12$$
$$= 6 - 5\lambda + \lambda^2 - 12$$
$$= \lambda^2 - 5\lambda - 6$$
$$= (\lambda - 6)(\lambda + 1)$$

Then $\lambda = 6$ or $\lambda = -1$ will make the determinat 0.

11 A & B are 2×2 matrices.

a) $\det(A+B) \neq \det(A) + \det(B)$ in general!

Ex: Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Then $\det(A) = 0$, $\det(B) = 0$

but $\det(A+B) = 1$.

In general let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$; $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$
$$\det(B) = b_{11}b_{22} - b_{12}b_{21}$$

$$\det(A+B) = \det\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

$$= (a_{11}+b_{11})(a_{22}+b_{22}) - (a_{12}+b_{12})(a_{21}+b_{21})$$

$$= (a_{11}a_{22} + b_{11}b_{22} + a_{11}b_{22} + b_{11}a_{22}) - (a_{12}a_{21} + b_{12}b_{21} + a_{12}b_{21} + b_{12}a_{21})$$

Note that we get some extra terms!

(b) $\det(AB) = \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix}$

$$= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$$

$$= a_{11}b_{11}a_{22}b_{22} + \cancel{a_{11}b_{11}a_{21}b_{12}} + a_{12}b_{21}a_{21}b_{12} + \cancel{a_{12}b_{21}a_{22}b_{22}}$$

$$- \cancel{a_{11}b_{12}a_{21}b_{11}} - a_{11}b_{12}a_{22}b_{21} - a_{12}b_{22}a_{21}b_{11} - \cancel{a_{12}b_{22}a_{22}b_{21}}$$

$$= a_{11}b_{11}a_{22}b_{22} - a_{12}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21}$$

$$\det(A)\det(B) = (a_{11}a_{22} - a_{21}a_{12})(b_{11}b_{22} - b_{12}b_{21})$$

$$= a_{11}a_{22}b_{11}b_{22} - a_{12}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{21}a_{12}b_{12}b_{21}$$

$\therefore \det(AB) = \det(A)\det(B)$

(c) Similarly you can show that $\det(BA) = \det(AB)$

or:

$\det(AB) = \det(A)\det(B)$

$\det(BA) = \det(B)\det(A)$

but $\det(A)\det(B) = \det(B)\det(A)$

$\therefore \det(AB) = \det(BA)$

§2.2

1. (a) $\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} = -24$

(b) $\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 5 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & 1 & 1 \\ -1 & -1 & -1 & 2 \end{vmatrix}$

$= 30$

(c) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1$

4 set det $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{pmatrix} = 0$

Then $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 8 & c-1 \\ 1 & c & 3 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 8 & c-1 \\ 0 & c-1 & 2 \end{vmatrix}$

$= 1 \cdot (16 - (c-1)^2)$

$= 16 - (c^2 - 2c - 1)$

$= 0$ given.

$\Rightarrow c^2 - 2c - 15 = 0$

$(c-5)(c+3) = 0$

$\therefore$ $c = 5$ or $c = -3$ will make the determinat zero.

(5) $\det(\alpha A) = \alpha^n \det(A)$; A is $n \times n$.

$$\det(\alpha A) = \begin{vmatrix} \alpha a_{11} & \cdots & \alpha a_{1n} \\ \vdots & \ddots & \vdots \\ \alpha a_{n1} & \cdots & \alpha a_{nn} \end{vmatrix}$$

$$= \alpha \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{n1} & \alpha a_{n2} & \cdots & \alpha a_{nn} \end{vmatrix}$$ factoring an α from the first row

$$= \alpha^2 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \alpha a_{31} & \alpha a_{32} & \cdots & \alpha a_{3n} \\ \vdots & & \ddots & \\ \alpha a_{n1} & \alpha a_{n2} & \cdots & \alpha a_{nn} \end{vmatrix}$$ factoring an α from the 2^{nd} row

$\vdots$

$$= \alpha^n \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$ factoring an α from the n^{th} row

$\det(\alpha A) = \alpha^n \det(A)$.

12

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

$$\det(V) = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2-x_1 & x_2^2-x_1^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2-x_1 & x_2^2-x_1^2 \\ 0 & x_3-x_1 & x_3^2-x_1^2 \end{vmatrix}$$

$$= (x_2-x_1)(x_3^2-x_1^2)-(x_3-x_1)(x_2^2-x_1^2)$$

$$=(x_2-x_1)(x_3-x_1)(x_3+x_1)-(x_3-x_1)(x_2-x_1)(x_2+x_1)$$

$$=(x_2-x_1)(x_3-x_1)(x_3+x_1-x_2+x_1)$$

$$=(x_2-x_1)(x_3-x_1)(x_3-x_2).$$

14 $\det(AB) = \det(A)\det(B)$.

$\therefore$ If $\det(A) = 0$ or $\det(B) = 0$

or if both $\det(A) = 0$ & $\det(B) = 0$

then $\det(AB) = 0$

$\therefore$ For $\det(AB) \neq 0$, we need to

have $\det(A) \neq 0$ and $\det(B) \neq 0$.

$\therefore$ AB non singular iff both A and

B are non singular.

§2.3

1 (c) $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$

$$\det(A) = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 0 & 3 & 0 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= 3$$

$A_{11} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$; $A_{12} = -\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} = 0$

$A_{13} = \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} = 6$; $A_{21} = -\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = 5$

$A_{22} = \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} = 1$; $A_{23} = -\begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} = -8$

$A_{31} = \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2$; $A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1$

$A_{33} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5$

$$\therefore \quad A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \frac{1}{3} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -1 & 5/3 & 2/3 \\ 0 & 1/3 & 1/3 \\ 2 & -8/3 & -5/3 \end{pmatrix}$$

2 (c)

$$\left.\begin{aligned} 2x_1 + x_2 - 3x_3 &= 0 \\ 4x_1 + 5x_2 + x_3 &= 8 \\ -2x_1 - x_2 + 4x_3 &= 2 \end{aligned}\right\} \quad \begin{pmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ -2 & -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 2 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -3 \\ 0 & 3 & 7 \\ 0 & 0 & 1 \end{vmatrix} = 6$$

$$\det(A_1) = \begin{vmatrix} 0 & 1 & -3 \\ 8 & 5 & 1 \\ 2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -3 \\ 8 & 5 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -3 \\ 0 & 5 & -3 \\ 2 & 0 & 1 \end{vmatrix} = 24$$

$$\det(A_2) = \begin{vmatrix} 2 & 0 & -3 \\ 4 & 8 & 1 \\ -2 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -3 \\ 0 & 8 & 7 \\ 0 & 2 & 1 \end{vmatrix} = -12$$

$$\det(A_3) = \begin{vmatrix} 2 & 1 & 0 \\ 4 & 5 & 8 \\ -2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -3 \\ 0 & 3 & 8 \\ 0 & 0 & 2 \end{vmatrix} = 12$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{24}{6} = 4$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-12}{6} = -2$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{12}{6} = 2.$$

(3) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

Recall that, $A^{-1} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \cdot \dfrac{1}{\det(A)}$

$\therefore$ (2,3) entry of A^{-1} is $\dfrac{A_{32}}{\det(A)}$.

$A_{32} = -\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = -3$

$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{vmatrix} = 4.$

$\therefore$ (2,3) entry of $A^{-1} = \dfrac{-3}{4}$

5 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

(a) $\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$= 0$

(b) $A_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} = -1$; $A_{12} = -\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 2$

$A_{13} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1$; $A_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2$

$A_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = -4$; $A_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2$

$A_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1$; $A_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 2$

$A_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$

$\therefore \text{Adj}(A) = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}$

$A \cdot \text{Adj}(A) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$