Math 2360: Linear Algebra
Fall 2009 - Section 004
Final Exam - Answer Book

- **Exam time: 7.30 am - 10.00 am (2 hours and 30 minutes)**
- This is a closed book exam.
- Answer all ten (10) questions.
- Show all the necessary work to earn full credit.
- Answers written on the test paper will not be graded.
- Please print your name on the first page of your answer book.
- Please write your answers clearly within the space provided in the answer book.

Name: KEY

(1) Solve the following pair of equations in 4 variables:

$$\begin{aligned} x_1 + 2x_3 - x_4 &= 2 \\ 2x_1 + x_2 + 2x_4 &= 1 \end{aligned}$$

Use either echelon form or reduced row echelon form.

Writing the matrix form:

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 2 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 2 & 1 & 0 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 - 2R_1 \to R_2} \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & -4 & 4 & -3 \end{array}\right)$$

This is the reduced row echelon form.

Set $x_3 = u$, $x_4 = v$ and we get

$$x_1 = 2 - 2u + v$$

$$x_2 = -3 + 4u - 4v$$

(2) Let

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$$

Calculate AB and BA and find out if they are equal.

$$AB = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 4 \\ 2 & 2 & 2 \\ 4 & 0 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & 9 \\ 3 & 2 & 0 \end{pmatrix}$$

$$\therefore \quad AB \neq BA$$

(3) Let A be the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}$$

Calculate A^{-1} of this matrix.

Method 1:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 2R_1 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 + R_2 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 1 \end{array}\right) \xrightarrow{R_2/2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & -2 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 - R_2 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & -2 & 1 & 1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \end{array}\right)$$

$$\therefore \; A^{-1} = \begin{pmatrix} 1 & -1/2 & 0 \\ -2 & 1 & 1 \\ 0 & 1/2 & 0 \end{pmatrix}$$

Method 2:

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \frac{1}{-2} \begin{pmatrix} -2 & 4 & 0 \\ 1 & -2 & -1 \\ 0 & -2 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & -2 & 0 \\ -1/2 & 1 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}^T$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1/2 & 0 \\ -2 & 1 & 1 \\ 0 & 1/2 & 0 \end{pmatrix}.$$

(4) Find a basis for the null space of the following matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

Set $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

i.e. $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_3 - 2R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right)$

$\xrightarrow{R_3 + 3R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 - 2R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\therefore$ Set $x_3 = u$, then,

$x_1 = u$

$x_2 = -u$

i.e. $\begin{pmatrix} u \\ -u \\ u \end{pmatrix}$ is the null space of A

$\therefore$ The basis $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ spans the null space of A

(5) For the matrix in problem 4, check
   (a) if the rows are linearly independant
   (b) if yes, go to the next problem; if not, find which rows are linearly independant.

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a) $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$, Let R_1; R_2, R_3 denote the 1^{st}, 2^{nd} & 3^{rd} rows of A.

To check linear independance, set

$$a_1 R_1 + a_2 R_2 + a_3 R_3 = (0 \ \ 0 \ \ 0)$$

i.e $(a_1 \ \ 2a_1 \ \ a_1) + (0 \ \ a_2 \ \ a_2) + (2a_3 \ \ a_3 \ \ -a_3) = (0 \ \ 0 \ \ 0)$

$\therefore$
$$\left.\begin{aligned} a_1 + 2a_3 &= 0 \\ 2a_1 + a_2 + a_3 &= 0 \\ a_1 + a_2 - a_3 &= 0 \end{aligned}\right\} \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow[R_3 - R_1 \to R_1]{R_2 - 2R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 - R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore$ We can set $a_3 = u$, and we get

$$a_1 = -2u, \quad a_2 = 3u$$

For example we can use $u = 1$, and get

$$a_1 = -2, \quad a_2 = 3, \quad a_3 = 1.$$

$\therefore$ Rows are not linearly independant.

b) We may pick row 1 & 2 to be linearly independant (note that we can write one row using the other two eg: $R_3 = 2R_1 - 3R_2$)

(6) Consider the following vectors in $\mathbb{R}^5$

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(a) Are the vectors v_1, v_2 and v_3 linearly independant ?
(b) Find a vector which is not in span $\{v_1, v_2, v_3\}$.

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a) Set $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$

i.e $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right) \xrightarrow[R_5 - 2R_1 \to R_5]{R_4 - 3R_1 \to R_4} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow[R_5 - R_2 \to R_5]{R_4 - 2R_2 \to R_4} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R_5 - R_3 \to R_5]{R_4 - R_3 \to R_4} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore a_1 = a_2 = a_3 = 0$.

$\therefore v_1, v_2$ & v_3 are linearly independant.

b) $\text{Span}\{v_1, v_2, v_3\} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ 3a_1 + 2a_2 + a_3 \\ 2a_1 + a_2 + a_3 \end{pmatrix}$

For example, if we take $a_1 = a_2 = a_3 = 0$, it will force $3a_1 + 2a_2 + a_3 = 0$ and $2a_1 + a_2 + a_3 = 0$.

$\therefore$ we cannot have $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ in span $\{v_1, v_2, v_3\}$.

(7) Calculate the determinant of the following matrix A:

$$A = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 3 & 4 & 1 & 7 \\ -2 & 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

$$\det A = -4 \cdot \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{vmatrix} = -4 \cdot 1 \cdot \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = -4 \cdot 1 \cdot (-1+6)$$

$$\det = -20$$

(8) Let A be the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(a) Write down $\det A$.
(b) Write down trace A.
(c) Write down $I + A$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
(d) Write down $\det(I + A)$.
(e) Verify if it is true that $\det(I + A) = 1 + \text{trace } A + \det A$.

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a) $\det A = ad - bc$

b) $\text{trace } A = a + d$

c) $I + A = \begin{pmatrix} a+1 & b \\ c & d+1 \end{pmatrix}$

d) $\det(I + A) = (a+1)(d+1) - bc$

e) $\det(I+A) = ad + a + d + 1 - bc$

$= (ad - bc) + a + d + 1$

$= 1 + (a+d) + (ad - bc)$

$= 1 + \text{trace}(A) + \det(A)$.

(9) Using the matrix A in problem 7. consider the linear equation

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 3 & 4 & 1 & 7 \\ -2 & 0 & 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Calculate x_2 using Cramer's rule.

In 7 we found that $\det A \neq 0$

$$x_2 = \frac{\begin{vmatrix} 1 & 0 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 7 \\ -2 & 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 3 & 4 & 1 & 7 \\ -2 & 0 & 0 & -1 \end{vmatrix}} = \frac{-1 \cdot \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ -2 & 0 & -1 \end{vmatrix}}{-4 \cdot \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ -2 & 0 & -1 \end{vmatrix}} = \frac{1}{4}.$$

(10) Let L be a map from $\mathbb{R}^3$ to $\mathbb{R}^3$ described as follows:

$$L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix}$$

(a) Is L a linear transformation ?
(b) Calculate $L(L(x))$.
(c) Find a matrix A such that $L(x) = Ax$.

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a) Let $\alpha, \beta \in \mathbb{R}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$\alpha x + \beta y = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \end{pmatrix}$$

$$\therefore L(\alpha x + \beta y) = \begin{pmatrix} \alpha x_3 + \beta y_3 \\ \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{pmatrix} = \alpha \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix} + \beta \begin{pmatrix} y_3 \\ y_1 \\ y_2 \end{pmatrix}$$

Note that $L(x) = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix}$ & $L(y) = \begin{pmatrix} y_3 \\ y_1 \\ y_2 \end{pmatrix}$

$$\therefore L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$$

$\therefore$ L is a linear transformation.

b) $$L(L(x)) = L \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

c) For this, look at what L does on the natural basis in $\mathbb{R}^3$, $L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Note that $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix}$.