Answers to

Midterm II

(Make up).

① $3 \det\begin{pmatrix} 1 & 9 \\ 4 & 6 \end{pmatrix} - 5 \det\begin{pmatrix} 8 & 9 \\ 2 & 6 \end{pmatrix}$

$+ 7 \det\begin{pmatrix} 8 & 1 \\ 2 & 4 \end{pmatrix}$

$= 3\,[6-36] - 5\,[48-18] + 7\,[32-2]$

$= -3 \times 30 - 5 \times 30 + 7\,[30]$

$= (-8+7) \times 30$

$= -30$

② $\alpha \begin{pmatrix} 1 \\ 2 \\ 5 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 3 \\ 6 \\ 7 \end{pmatrix} + \gamma \begin{pmatrix} 4 \\ 5 \\ 8 \\ 23 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\alpha + 2\beta + 4\gamma = 0$

$2\alpha + 3\beta + 5\gamma = 0$

$5\alpha + 6\beta + 8\gamma = 0$

$-\alpha + 7\beta + 23\gamma = 0$

Aug Matrix is

$$\begin{pmatrix} 1 & 2 & 4 & 0 \\ 2 & 3 & 5 & 0 \\ 5 & 6 & 8 & 0 \\ -1 & 7 & 23 & 0 \end{pmatrix}$$

$$\longleftrightarrow \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & -4 & -12 & 0 \\ 0 & 9 & 27 & 0 \end{pmatrix}$$

$$\longleftrightarrow \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha + 2\beta + 4\gamma = 0$$
$$\beta + 3\gamma = 0$$

$$\beta = -3\gamma$$

$$\alpha = -2\beta - 4\gamma$$

$$= -2[-3\gamma] - 4\gamma$$

$$= 6\gamma - 4\gamma = 2\gamma$$

If $\gamma = t$, $\beta = -3t$, $\alpha = 2t$

$$\therefore \boxed{\begin{aligned} \alpha &= 2t \\ \beta &= -3t \\ \gamma &= t \end{aligned}} \leftarrow \text{sol}^{n}.$$

$\because (0, 0, 0)$ is not the only solution

$\therefore$ vectors are linearly dependent.

3

$$\begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & -1 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Aug matrix is.

$$\begin{pmatrix} 3 & 4 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & 4 & 1 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 6 & 8 & 2 & 0 \\ 6 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 6 & 8 & 2 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 3 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x - z &= 0 \\ y + z &= 0 \end{aligned} \implies \begin{aligned} x &= z \\ y &= -z \end{aligned}$$

If $z = t$, $x = t$, $y = -t$.

Null space is given by

$$\begin{pmatrix} t \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\text{Null space} = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Basis

(4) From problem (3) we know that

$\begin{pmatrix} t \\ -t \\ t \end{pmatrix}$ is in the null space of the matrix in (3).

For $t=1$ we have

$$\begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & -1 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 1\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} - 1\begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}$$

Column space is

$$\text{span}\left\{\begin{pmatrix}3\\2\\3\end{pmatrix}, \begin{pmatrix}4\\1\\4\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}$$

$$= \text{span}\left\{\begin{pmatrix}3\\2\\3\end{pmatrix}, \begin{pmatrix}4\\1\\4\end{pmatrix}\right\}$$

$\because$ third vector is the linear combination of the first two.

Since $\begin{pmatrix}3\\2\\3\end{pmatrix}, \begin{pmatrix}4\\1\\4\end{pmatrix}$ are linearly independent

they form a basis of the column space

A Basis of the column space $= \left\{\begin{pmatrix}3\\2\\3\end{pmatrix}, \begin{pmatrix}4\\1\\4\end{pmatrix}\right\}$.