Sol$^{\underline{n}}$ to Mid Term II

Math 2360.

①

① writing

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

we get

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 0 & -2 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Augmented Matrix

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & -2 & 0 \\ -1 & 1 & 2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 3 & 3 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha - \gamma = 0 \qquad \Rightarrow \quad \alpha = \gamma$$
$$\beta + \gamma = 0 \qquad \quad \beta = -\gamma$$

if $\gamma = t$, $\alpha = t$, $\beta = -t$.

$\therefore$ $\alpha = t$, $\beta = -t$, $\gamma = t$ for any real number t would be a solution.

vectors are linearly dependent

—— x ——

2 (a) $A^3 = A$

$\Rightarrow \det(A^3) = \det A$

But $\det A^3 = (\det A)^3$

Hence $(\det A)^3 = \det A$

$\Rightarrow (\det A)^3 - \det A = 0$

$\Rightarrow \det A\left[(\det A)^2 - 1\right] = 0$

$\Rightarrow \det A\,[\det A + 1][\det A - 1] = 0$

$\Rightarrow$ either $\det A = 0$. or $\det A = -1$ or $\det A = 1$

2 b

$$B = R\,A\,R^T$$

$$\Rightarrow \det B = \det\left[R\,A\,R^T\right]$$

$$= \left[\det R\right]\left[\det A\right]\left[\det R^T\right] \quad (*)$$

But $R R^T = I$

$$\Rightarrow \det(R R^T) = \det I = 1$$

$$\Rightarrow (\det R)(\det R^T) = 1$$

$$\Rightarrow (\det R^T) = \frac{1}{\det R} \quad \text{since } \det R \neq 0$$

$$(**)$$

from $(*)$ & $(**)$ we have

$$\det B = \left[\det R\right]\left[\det A\right]\frac{1}{\det R}$$

$$= \det A.$$

3 (a) Let v_1 and v_2 be two elements in V.

writing

$$v_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad v_2 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

we have

$$v_1 + v_2 = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{pmatrix} \quad \text{and } \alpha v_1 = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \\ \alpha x_4 \end{pmatrix}$$

where α is a scalar.

$\therefore$ v_1 and $v_2 \in V$ it follows that

$$x_1 + x_2 + x_3 = 0$$
$$y_1 + y_2 + y_3 = 0$$

It follows that

$$(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) =$$
$$(x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = 0 + 0$$
$$= 0.$$

Hence $v_1 + v_2 \in V$.

Furthermore

$$\alpha x_1 + \alpha x_2 + \alpha x_3 =$$
$$\alpha(x_1 + x_2 + x_3) = \alpha \cdot 0 = 0 .$$

Hence

$$\alpha v_1 \in V .$$

It follows that V is closed under addition and scalar multiplication.

But $V \subset \mathbb{R}^4$ which is already a vector space

Hence V is a vector space.

3(b) Take

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Clearly $v_1 \in U$ and $v_2 \in U$ because $|1| = |1|$ & $|1| = |-1|$.

$$v_1 + v_2 = \begin{pmatrix} 1+1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

The vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ does not belong to U.

$v_1 + v_2 \notin U$.

Hence U is not closed under addition.

Hence U is not a vector space.

Remark: Actually U is closed under scalar multiplication. Can you show it?

4 Let $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ be in the null space

It follows that

$$x_1 + 2x_2 + 3x_3 + x_4 = 0$$
$$x_2 + x_3 + x_4 = 0.$$
$$x_2 + x_3 + x_4 = 0.$$
$$4x_1 - x_2 + 3x_3 - 5x_4 = 0$$

Augmented matrix.

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 4 & -1 & 3 & -5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -9 & -9 & -9 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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It follows that

$$x_1 + x_3 - x_4 = 0$$
$$x_2 + x_3 + x_4 = 0$$

$$\Rightarrow \boxed{\begin{aligned} x_1 &= x_4 - x_3 \\ x_2 &= -x_4 - x_3 \end{aligned}}$$

If $x_3 = t$, $x_4 = s$ we have

$$\boxed{\begin{aligned} x_1 &= s - t \\ x_2 &= -s - t \end{aligned}}$$

$\therefore$ Null space is given by

$$\begin{pmatrix} s-t \\ -s-t \\ t \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} s + \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} t$$

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Null space is the span of vectors.

$$\text{span}\left\{\begin{pmatrix}1\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}\right\}$$