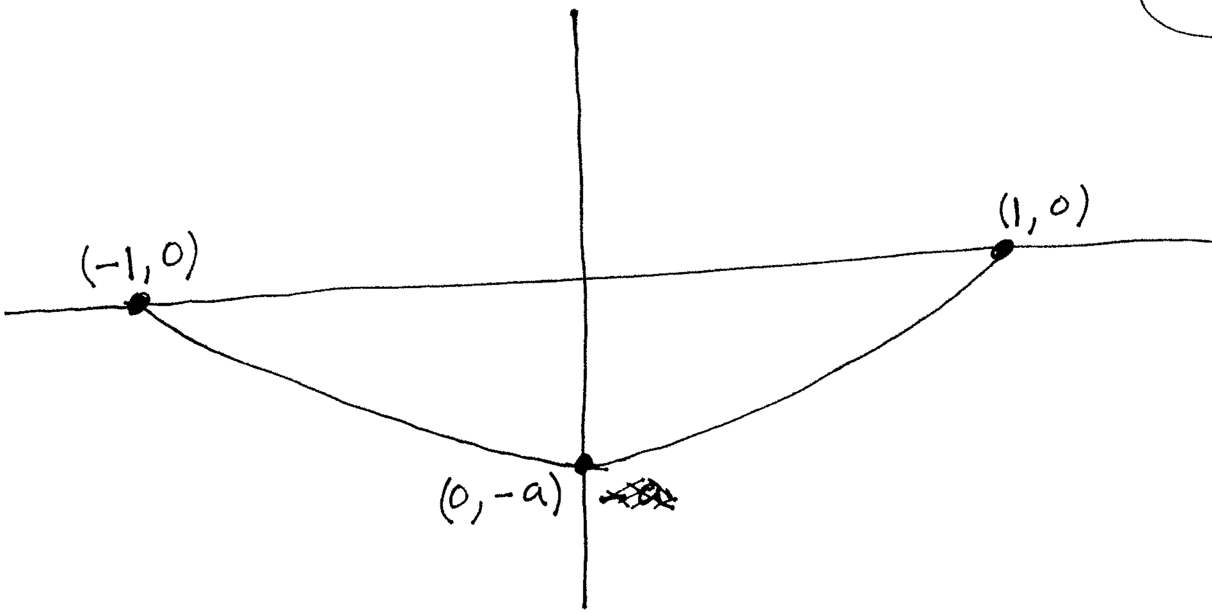
The story of football
Volume and Surface area
calculations.

Example:

Consider a curve

$$y = a(x^2 - 1), \quad -1 \le x \le 1, \quad a > 0.$$



Arc Length calculation

$$\int_{-1}^{1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = 2ax, \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + 4a^2x^2}$$

$$\text{Arc Length} = \int_{-1}^{1} \sqrt{1 + 4a^2x^2}\, dx$$

Define

$$\tan\theta = 2ax$$

$$\underbrace{1+\tan^2\theta}_{\sec^2\theta} = 1 + 4a^2x^2$$

$$\Rightarrow \sqrt{1+4a^2x^2} = \sec\theta$$

$$2a\,dx = \sec^2\theta\, d\theta$$

$$\text{Arc Length} = \int_{x=-1}^{x=1} \sec\theta \, \frac{\sec^2\theta}{2a}\, d\theta$$

$$= \frac{1}{2a}\int_{x=-1}^{x=1} \sec^3\theta\, d\theta$$

From the integration table in Appendix D formula 427 and 428 we write

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$$\int \sec^3\theta \, d\theta =$$

$$\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

writing

$$\left.\begin{aligned} \sec\theta &= \sqrt{1+4a^2x^2} \\ \tan\theta &= 2ax \end{aligned}\right\}$$ we obtain

Arc length =

$$\frac{1}{2a}\left[\frac{1}{\cancel{2}}\,\cancel{2}ax\sqrt{1+4a^2x^2} + \frac{1}{2}\ln\left|\sqrt{1+4a^2x^2}+2ax\right|\right]_{-1}^{1}$$

$$= \frac{1}{2a}\left[ax\sqrt{1+4a^2x^2} + \frac{1}{2}\ln\left|\sqrt{1+4a^2x^2}+2ax\right|\right]_{-1}^{1}$$

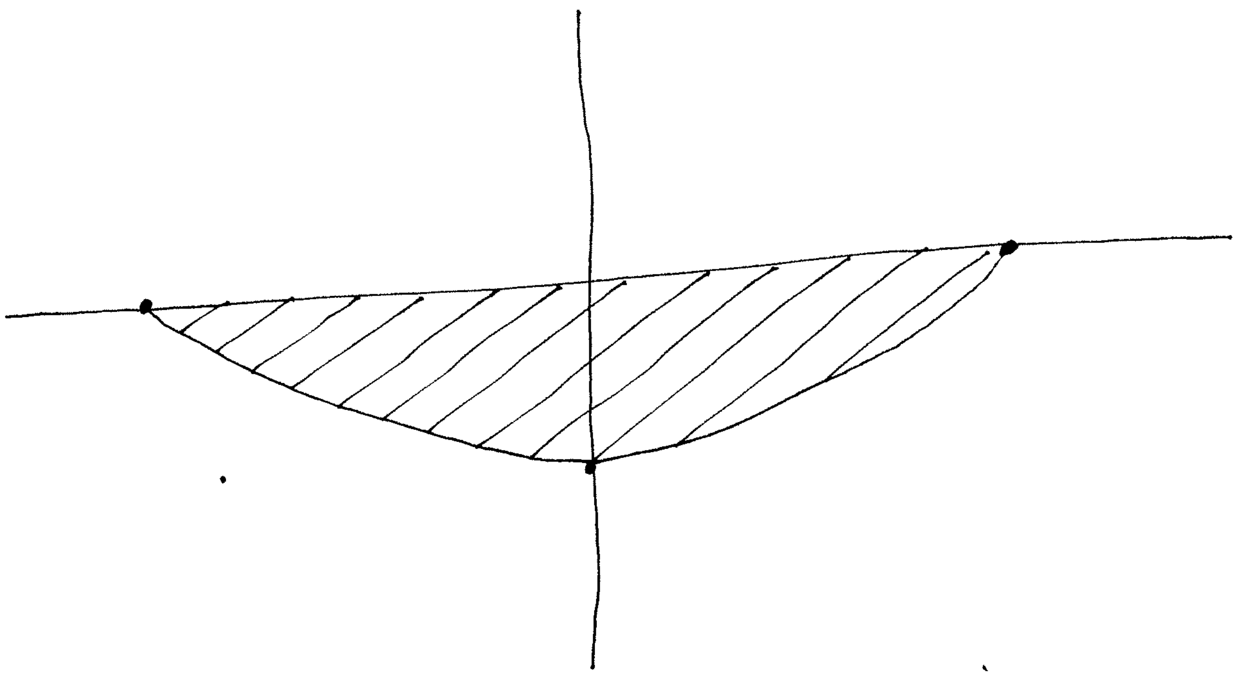
$$= \frac{1}{2a}\left[2a\sqrt{1+4a^2} + \frac{1}{2}\ln\left|\frac{\sqrt{1+4a^2}+2a}{\sqrt{1+4a^2}-2a}\right|\right]$$

It would follow that the arc length is given by

Arc length =

$$\boxed{\sqrt{1+4a^2} + \frac{1}{4a}\ln\left|\frac{\sqrt{1+4a^2}+2a}{\sqrt{1+4a^2}-2a}\right|}$$

Area calculation:



Area A of the shaded region

$$= \int_{-1}^{1} -f(x)\,dx = \int_{-1}^{1} a(1-x^2)\,dx$$

$$= a\left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

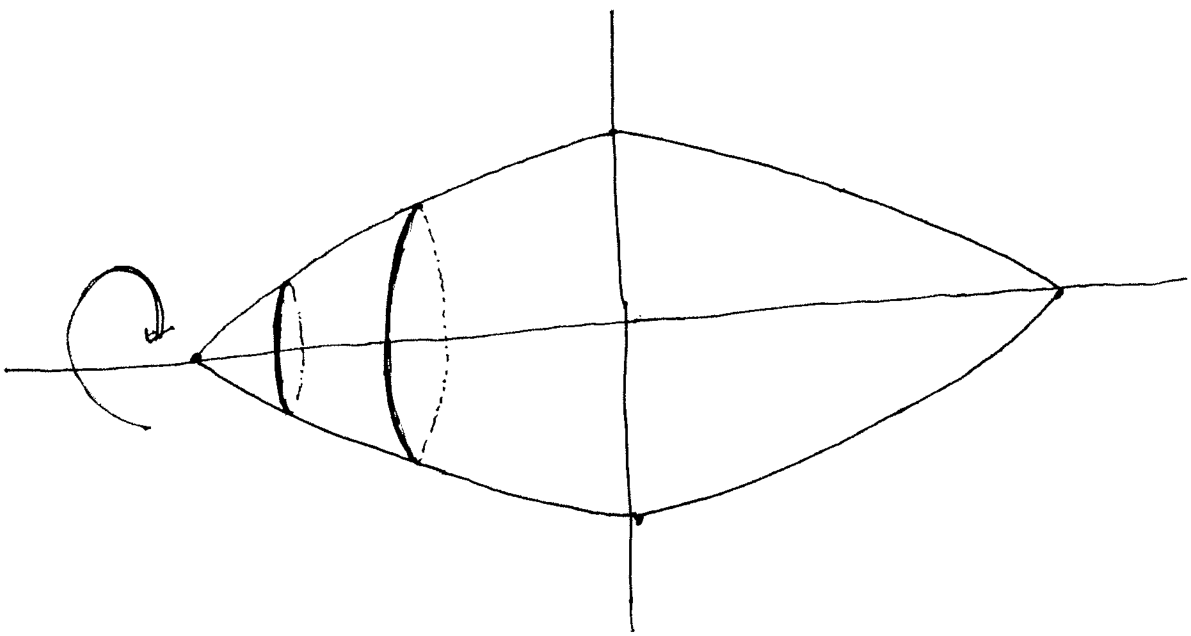
$$= a\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right]$$

$$= a\left[2 - \frac{2}{3}\right]$$

$$= a\,\frac{6-2}{3} = \frac{4}{3}a$$

x

We now consider the curve (A) rotated about the x-axis



The solid object generated has the shape of a football.

We want to calculate the surface area and volume of a football.

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Surface area S is given by

$$S = \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_{-1}^{1} 2\pi a(x^2-1) \sqrt{1 + 4a^2x^2} \, dx$$

Substituting

$$x = \frac{1}{2a} \tan\theta .$$

we obtain

$$S = \int_{-1}^{1} \cancel{2}\pi \cancel{a} \left(\frac{1}{4a^2} \tan^2\theta - 1\right) \sec\theta \cdot \frac{1}{\cancel{2}\cancel{a}} \sec^2\theta \, d\theta .$$

$$= \int_{-1}^{1} \pi \left(\frac{1}{4a^2} \tan^2\theta - 1\right) \sec^3\theta \, d\theta .$$

$$\tan^2\theta = \sec^2\theta - 1$$

$$\frac{1}{4a^2}\tan^2\theta = \frac{1}{4a^2}\sec^2\theta - \frac{1}{4a^2}$$

$$\frac{1}{4a^2}\tan^2\theta - 1 = \frac{1}{4a^2}\sec^2\theta - \left(\frac{1}{4a^2} + 1\right)$$

$$\pi\left(\frac{1}{4a^2}\tan^2\theta - 1\right)\sec^3\theta = .$$

$$\frac{\pi}{4a^2}\sec^5\theta - \frac{1+4a^2}{4a^2}\pi\sec^3\theta .$$

$$S = \frac{\pi}{4a^2}\int_{x=-1}^{x=1}\sec^5\theta\, d\theta$$

$$- \frac{1+4a^2}{4a^2}\pi\int_{x=-1}^{x=1}\sec^3\theta\, d\theta$$

$$\int \sec^5\theta \, d\theta$$

$$= \frac{\sec^3\theta \tan\theta}{4} + \frac{3}{4}\int \sec^3\theta \, d\theta .$$

Reduction formula in appendix D

$$\frac{\pi}{4a^2}\int \sec^5\theta \, d\theta$$

$$= \frac{\pi}{16a^2}\sec^3\theta \tan\theta + \frac{3\pi}{16a^2}\int \sec^3\theta \, d\theta$$

It follows that

$$S = \frac{\pi}{16a^2}\sec^3\theta \tan\theta \Big|_{x=-1}^{x=1} + \underbrace{\left[\frac{3\pi}{16a^2} - \frac{1+4a^2}{4a^2}\pi\right]}_{\parallel} \int_{x=-1}^{x=1} \sec^3\theta \, d\theta$$

$$\frac{3\pi - 4\pi - 16a^2\pi}{16a^2}$$

$$= -\left(\frac{1+16a^2}{16a^2}\right)\pi$$

$$S = \left\{ \frac{\pi}{16a^2} \sec^3\theta \tan\theta - \left(\frac{1}{16a^2} + 1\right) \frac{\pi}{2} \left[\cancel{\frac{1}{2}} \sec\theta \tan\theta + \cancel{\frac{1}{2}} \ln|\sec\theta + \tan\theta| \right] \right\}_{x=-1}^{x=1}$$

$$= \left[\frac{\pi}{\underset{8}{\cancel{16}}a^{\cancel{2}}} (1+4a^2x^2)^{3/2} \cancel{2}\cancel{a}x - \frac{\pi}{2}\left(1 + \frac{1}{16a^2}\right) \left[2ax(1+4a^2x^2)^{1/2} + \ln\left|(1+4a^2x^2)^{1/2} + 2ax\right| \right]_{x=-1}^{1}\right.$$

$$= \frac{\pi}{\cancel{8}4a}(1+4a^2)^{3/2} - \frac{\pi}{\cancel{4}}\left(1+\frac{1}{16a^2}\right) 2a(1+4a^2)^{1/2} + \ln\left|\frac{(1+4a^2)^{1/2} + 2a}{(1+4a^2)^{1/2} - 2a}\right|$$

$$= \frac{\pi}{4a}(1+4a^2)^{3/2} - 2\pi a\left(1 + \frac{1}{16a^2}\right)(1+4a^2)^{1/2} + \ln\left|\frac{(1+4a^2)^{1/2} + 2a}{(1+4a^2)^{1/2} - 2a}\right|$$

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Volume V is given by

$$V = \int_{-1}^{1} \pi y^2 \, dx \quad \leftarrow \text{disk method.}$$

$$= \int_{-1}^{1} \pi a^2 (x^2 - 1)^2 \, dx$$

$$= \pi a^2 \int_{-1}^{1} (x^4 - 2x^2 + 1) \, dx$$

$$= \pi a^2 \left[\frac{x^5}{5} - \frac{2}{3} x^3 + x \right]_{-1}^{1}$$

$$= \pi a^2 \left[\left(\frac{1}{5} - \frac{2}{3} + 1 \right) - \left(-\frac{1}{5} + \frac{2}{3} - 1 \right) \right]$$

$$= \pi a^2 \left[\frac{2}{5} - \frac{4}{3} + 2 \right]$$

$$= \frac{6 - 20 + 30}{15} \pi a^2 = \frac{16}{15} \pi a^2$$