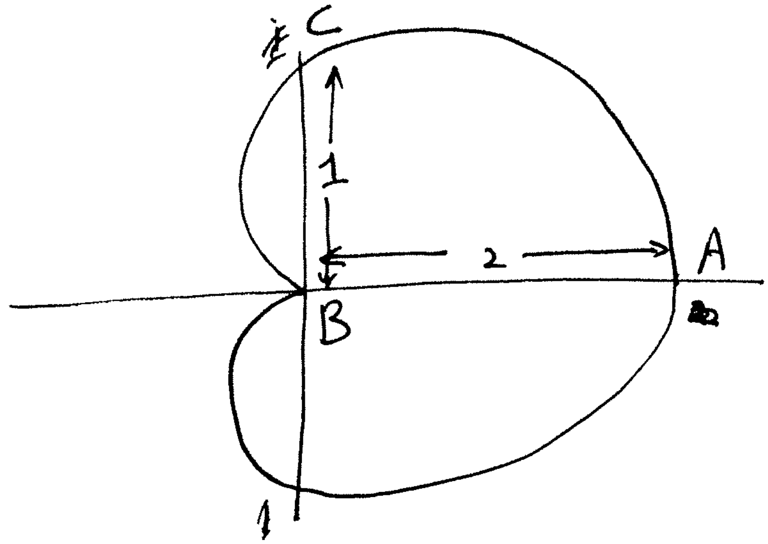
Area and circumference of a cardiod.

A cardiod (Arc length calculations)



$r = 1 + \cos\theta \leftarrow$ eqn of a cardiod.

$$\frac{dr}{d\theta} = -\sin\theta$$

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (1+\cos\theta)^2 + (-\sin\theta)^2 \\ &= 1 + 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta} \\ &= 1 + 2\cos\theta + 1 \\ &= 2 + 2\cos\theta \\ &= 2\left[1 + \cos\theta\right] \end{aligned}$$

Recall from trignometry

$$\cos 2y = 2\cos^2 y - 1$$

$\boxed{\text{plug in } 2y = \theta}$

$$\boxed{\cos\theta = 2\cos^2\frac{\theta}{2} - 1}$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 2(1+\cos\theta)$$

$$= 2 \cdot 2\cos^2\frac{\theta}{2}$$

$$= 4\cos^2\frac{\theta}{2}$$

$$\boxed{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2\cos\frac{\theta}{2}.}$$

Length of the arc ACB is given by

$$\int_{\theta=0}^{\theta=\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi} 2\cos\frac{\theta}{2}\, d\theta$$

$$= 4\sin\frac{\theta}{2}\Big|_0^{\pi}$$

$$= 4\sin\frac{\pi}{2} - 4\sin\frac{0}{2}$$

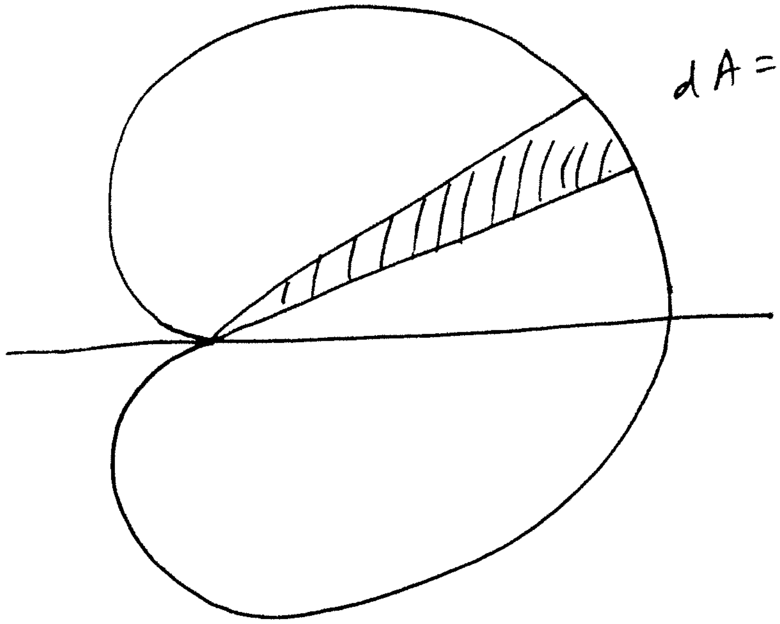
$$= 4$$

Circumference of the cardiod is.

$2 \times$ arc length of ACB

$= 2 \times 4 = 8$.

Area calculations



$dA = \frac{1}{2} r^2 \, d\theta$

$$\text{Area} = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$$

$$r^2 = (1+\cos\theta)^2 = 1 + 2\cos\theta + \cos^2\theta.$$
$$= 1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)$$
$$= \frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta$$

$$\text{Area} = \int_0^{2\pi} \left(\frac{3}{4} + \cos\theta + \frac{1}{4}\cos 2\theta\right) d\theta.$$
$$= \left[\frac{3}{4}\theta + \sin\theta + \frac{1}{8}\sin 2\theta\right]_0^{2\pi} = \frac{6\pi}{4}$$