Practice Problems for Midterm III

Ex 1

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \int_{x=0}^{x=1} \frac{-\sin\theta\, d\theta}{\sqrt{1-\cos^2\theta}} = \int_{\pi/2}^{0} -\, d\theta$$

$x = \cos\theta$

$dx = -\sin\theta . d\theta$

$$= \int_0^{\pi/2} d\theta = \pi/2$$

— x —

Ex 2

$$\int_0^\infty \frac{dx}{1+x^2} = \int_{x=0}^{x=\infty} \frac{\cancel{\sec^2}\theta\, d\theta}{\cancel{\sec^2\theta}} = \int_{\theta=0}^{\theta=\pi/2} d\theta$$

$$\left.\begin{aligned} x &= \tan\theta \\ dx &= \sec^2\theta\, d\theta \\ 1+x^2 &= 1+\tan^2\theta \\ &= \sec^2\theta \end{aligned}\right\}$$

$$= \pi/2$$

Ex:

$$\int_0^{\pi/2} \tan x\,dx = \int_{x=0}^{\pi/2} -\frac{du}{u} = -\ln u\Big|_{u=1}^{u=0} = \ln u\Big|_0^1$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$u = \cos x$$

$$du = -\sin x\,dx$$

$$\frac{du}{u} = -\tan x\,dx$$

$$\underset{N\to 0}{Lt}\ \ln 1 - \ln N$$

$$= -\underset{N\to 0}{Lt}\ \ln N.$$

$\boxed{\text{Does not exist}}$

Ex

$$\int_0^{\infty} \frac{dx}{1+e^x} = \int_{x=0}^{x=\infty} -\frac{du}{u} = \int_{u=2}^{u=1} -\frac{du}{u} = \int_{u=1}^{u=2} \frac{du}{u}$$

$$\frac{dx}{1+e^x} = \frac{e^{-x}\,dx}{e^{-x}+1} = \frac{-du}{u}$$

$$1+e^{-x} = u$$

$$du = -e^{-x}\,dx$$

$$= \ln u\Big|_1^2$$

$$= \ln 2 - \ln 1.$$

$$= \ln 2$$

Converges.

Ex: $\displaystyle\int_1^\infty \frac{dx}{x^3} = \left.\frac{x^{-3+1}}{-3+1}\right|_1^\infty = \left.-\frac{1}{2}\frac{1}{x^2}\right|_1^\infty$

$$= \frac{1}{2}$$

converges.

Ex $\displaystyle\int_0^\infty \frac{dx}{x^3} = \underset{\substack{b\to\infty \\ a\to 0}}{Lt}\int_a^b \frac{dx}{x^3} = \left.-\frac{1}{2}\frac{1}{x^2}\right|_a^b$

$$= -\frac{1}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

$$\underset{b\to\infty}{Lt}\ \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{1}{2a^2}$$

$$\underset{a\to 0}{Lt}\ \frac{1}{2a^2} \longrightarrow \infty$$ $\boxed{\text{Diverges}}$

Ex

$$\int_1^\infty \frac{dx}{1+x^3} \le \int_1^\infty \frac{dx}{1+x^2} = \int_{x=1}^{x=\infty} d\theta = \int_{\theta=\pi/4}^{\theta=\pi/2} d\theta$$

$x = \tan\theta$

$$= \frac{\pi}{4}.$$

when $x \geq 1$

$$x^2 \le x^3$$

$$\Rightarrow 1+x^2 \le 1+x^3$$

$$\Rightarrow \frac{1}{1+x^2} \geq \frac{1}{1+x^3}$$

Hence $\displaystyle\int_1^\infty \frac{dx}{1+x^3}$ converges.

Ex: $\sum_{n=3}^{\infty} \frac{1}{n \ln n} = \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} - - -$

$$= \sum_{m=1}^{\infty} \frac{1}{(m+2)\ln(m+2)}$$

$$\int_1^{\infty} \frac{1}{(x+2)\ln(x+2)} dx = \int_{x=1}^{x=\infty} \frac{du}{u} = \int_{\ln 3}^{\infty} \frac{du}{u}$$

$u = \ln(x+2)$

$du = \frac{1}{x+2} dx$

$$= \operatorname{Lt}_{a \to \infty} \int_{\ln 3}^{a} \frac{du}{u}$$

$$= \operatorname{Lt}_{a \to \infty} \ln a - \ln[\ln(3)]$$

Does not converge.

Ex:

$$x_n = \frac{n!}{n^n}$$

$$x_{n+1} = \frac{n+1\,!}{(n+1)^{n+1}}$$

$$\text{Ratio} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!}$$

$$= \cancel{\frac{(n+1)!}{n!}} \frac{n^n}{(n+1)^n \cancel{(n+1)}} = \frac{n^n}{(n+1)^n}$$

$$= \left(\frac{n}{n+1}\right)^n$$

$$= \left(\frac{1}{1+\frac{1}{n}}\right)^n$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\rho = \underset{n\to\infty}{Lt} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

Hence converges.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$$\int_1^{\infty} \frac{1}{1+x^2}\, dx = \frac{\pi}{4}.$$

Hence the sequence converges.

Ex $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n} \le \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} \frac{1}{1-\frac{1}{2}}.$

$= 1.$

$\boxed{\text{(sequence converges)}}$

Ex: $\sum_{n=1}^{\infty} \frac{10^n}{n!}$

$x_n = \frac{10^n}{n!}$

$x_{n+1} = \frac{10^{n+1}}{(n+1)!}$

$\text{ratio} = \frac{10^{n+1}}{(n+1)!} \frac{n!}{10^n} = \frac{\cancel{10^n}\cdot 10}{\cancel{10^n}} \frac{n!}{(n+1)!}$

$= 10/_{n+1}$; $\rho = \lim_{n\to\infty} \text{ratio} = 0$

$\boxed{\text{Converges}}$