Calculus II

Answers to Midterm II

1 Ans:

We take a representative element of area

$$dA = dA_1 - dA_2$$

where

$$dA_1 = \frac{1}{2} r_1^2 \, d\theta \; ; \; dA_2 = \frac{1}{2} r_2^2 \, d\theta$$

with

$$r_1 = a \quad \text{and} \quad r_2 = a(1-\cos\theta)$$

From the figure, we see that the required area

$$A = \int_{-\pi/2}^{\pi/2} dA_1 - dA_2$$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} (r_1^2 - r_2^2)\, d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} a^2 - a^2(1-\cos\theta)^2 \, d\theta$$

(2)

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} a^2 - a^2(1+\cos^2\theta - 2\cos\theta)\, d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[2a^2\cos\theta - a^2\cos^2\theta\right] d\theta$$

$$= \int a^2 \cos\theta\, d\theta - \int \frac{a^2}{2}\cos^2\theta\, d\theta$$

$$= \int a^2\cos\theta\, d\theta - \int \frac{a^2}{2}\,\frac{(1+\cos 2\theta)}{2}\, d\theta$$

$$= \int a^2\cos\theta\, d\theta - \int \frac{a^2}{4} d\theta - \int \frac{a^2}{4}\cos 2\theta\, d\theta$$

$$= a^2\sin\theta - \frac{a^2}{4}\theta - \frac{a^2}{4}\,\frac{\sin 2\theta}{2}\Bigg|_{-\pi/2}^{\pi/2}$$

$$= a^2 2 - \frac{a^2}{4}\pi - \frac{a^2}{8}\cdot 0$$

$$= a^2\left[2 - \frac{\pi}{4}\right] \longleftarrow \text{Ans.}$$

2) Ans:

sinx has odd power.

So we put sinx with dx and write

$$\int \cos^{2/3} x \sin^2 x \, [\sin x \, dx]$$

$$= \int \cos^{2/3} x \left(1 - \cos^2 x\right) [\sin x \, dx]$$

$\cancel{u = \sin x}$ $\quad u = \cos x$

$\cancel{du = \cos x}$ $\quad du = -\sin x \, dx$.

$$= -\int u^{2/3}(1-u^2)\,du$$

$$= \int -u^{2/3} + u^{8/3} \, du.$$

$$= \frac{-u^{5/3}}{5/3} + \frac{u^{11/3}}{11/3} \cancel{C} = \frac{3}{11} u^{11/3} - \frac{3}{5} u^{5/3} + C$$

$$= \frac{3}{11} \cos^{11/3} x - \frac{3}{5} \cos^{5/3} x + C.$$

3 Ans:

$$\int \frac{x+1}{\sqrt{4-x^2}}\,dx = \int \frac{x\,dx}{\sqrt{4-x^2}} + \int \frac{1}{\sqrt{4-x^2}}\,dx$$

$$\int \frac{x\,dx}{\sqrt{4-x^2}} =$$

$$u = 4 - x^2 \implies x\,dx = -\frac{1}{2}\,du$$
$$du = -2x\,dx$$

$$= \int \frac{-\frac{1}{2}\,du}{\sqrt{u}} = -\frac{1}{2}\int u^{-1/2}\,du$$

$$= -\cancel{\frac{1}{2}}\,\frac{u^{1/2}}{\cancel{1/2}} + C$$

$$= -\sqrt{u} + C$$

$$= -\sqrt{4-x^2} + C$$

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$$\int \frac{1}{\sqrt{4-x^2}}\,dx =$$

$$x = 2\sin\theta$$

$$dx = 2\cos\theta\,d\theta$$

$$4-x^2 = 4-4\sin^2\theta = 4\cos^2\theta$$

$$\sqrt{4-x^2} = 2\cos\theta$$

$$= \int \frac{1}{\cancel{2\cos\theta}}\,\cancel{2\cos\theta}\,d\theta = \int d\theta = \theta + C$$

$$= \sin^{-1}\frac{x}{2} + C$$

Hence $\displaystyle\int \frac{x+1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\frac{x}{2} - \sqrt{4-x^2} + C$

4 Ans:

$$x^2+4x-5=$$

$$(x+5)(x-1).$$

$$\therefore \frac{x}{x^2+4x-5} = \frac{A}{x+5} + \frac{B}{x-1}.$$

using the trick:

$$(x+5)\frac{x}{x^2+4x-5} = A + (x+5)\frac{B}{x-1}.$$

$$\Rightarrow \frac{x}{x-1} = A + (x+5)\frac{B}{x-1}.$$

sub. $x=-5$ we get

$$\frac{-5}{-6} = A \quad \Rightarrow A = 5/6.$$

$$(x-1)\frac{x}{x^2+4x-5} = (x-1)\frac{A}{x+5} + B.$$

$$\Rightarrow \frac{x}{x+5} = (x-1)\frac{A}{x+5} + B$$

Sub. $x=1$ we get

$$\boxed{B = 1/6.}$$

Thus

$$\frac{x}{x^2+4x-5} = \frac{5/6}{x+5} + \frac{1/6}{x-1}$$

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Alternatively

$$\frac{x}{x^2+4x-5} = \frac{A}{x+5} + \frac{B}{x-1}$$

$$= \frac{A(x-1)+B(x+5)}{(x+5)(x-1)}.$$

$$= \frac{(A+B)x+(5B-A)}{x^2+4x-5}$$

$\Rightarrow x = (A+B)x+(5B-A)$ for all x

comparing the co-efficients, we have

$A+B=1$; $5B-A=0$.

$\Rightarrow A=5B$, $5B+B=1 \Rightarrow B=1/6$, $A=5/6$.

5 Ans:

$$\int \underset{\mathrm{I}}{x^n} \underset{\mathrm{II}}{\sin x}\, dx$$

$$= x^n(-\cos x) - \int n x^{n-1}(-\cos x)\, dx.$$

$$= -x^n \cos x + n\int x^{n-1} \cos x\, dx.$$

$$\int \underset{\mathrm{I}}{x^{n-1}} \underset{\mathrm{II}}{\cos x}\, dx$$

$$= x^{n-1}\sin x - \int (n-1) x^{n-2} \sin x\, dx.$$

$$= x^{n-1}\sin x - (n-1)\int x^{n-2}\sin x\, dx$$

$$\therefore \int x^n \sin x\, dx = -x^n\cos x + n x^{n-1}\sin x$$
$$- n(n-1)\int x^{n-2}\sin x\, dx.$$

If

$$I_n = \int x^n \sin x \, dx$$

we have

$$I_n = -x^n \cos x + n x^{n-1} \sin x - n(n-1) I_{n-2}$$ where $n \geq 2$

For $n=2$ we have

$$I_2 = \int x^2 \sin x \, dx$$

$$= -x^2 \cos x + 2x \sin x - 2 I_0$$

where $I_0 = \int \sin x \, dx = -\cos x + C$

$$\therefore \int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2\cos x + C$$