$\boxed{\text{H. W. 8 (sol}^{n}\text{)}}$

1 write

$$\frac{1+x}{1-x} = z^2$$

$$\Rightarrow 1+x = (1-x)z^2 = z^2 - xz^2$$

$$\Rightarrow x + xz^2 = z^2 - 1$$

$$\Rightarrow x(1+z^2) = z^2 - 1$$

$$\Rightarrow x = \frac{z^2-1}{z^2+1}$$

$$\Rightarrow dx = \frac{(z^2+1)2z - (z^2-1)2z}{(z^2+1)^2}\, dz$$

$$= \frac{\cancel{2z^3} + 2z - \cancel{2z^3} + 2z}{(z^2+1)^2}\, dz$$

$$= \frac{4z\, dz}{(z^2+1)^2}$$

$$\sqrt{\frac{1+x}{1-x}}\,dx$$

$$= \frac{z \; 4z \; dz}{(z^2+1)^2}$$

$$= \frac{4z^2}{(z^2+1)^2}\,dz$$

got rid of the square root.

Let $z = \tan\theta$

$$z^2+1 = 1+\tan^2\theta = \sec^2\theta$$

$$dz = \sec^2\theta \, d\theta$$

$$\Rightarrow \frac{4z^2\,dz}{(z^2+1)^2} = \frac{4\tan^2\theta \cancel{\sec^2\theta}\,d\theta}{\cancel{\sec^4\theta}} \quad \sec^2\theta$$

$$= 4\frac{\sin^2\theta}{\cancel{\cos^2\theta}}\cancel{\cos^2\theta}\,d\theta$$

$$= 4\sin^2\theta\,d\theta.$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

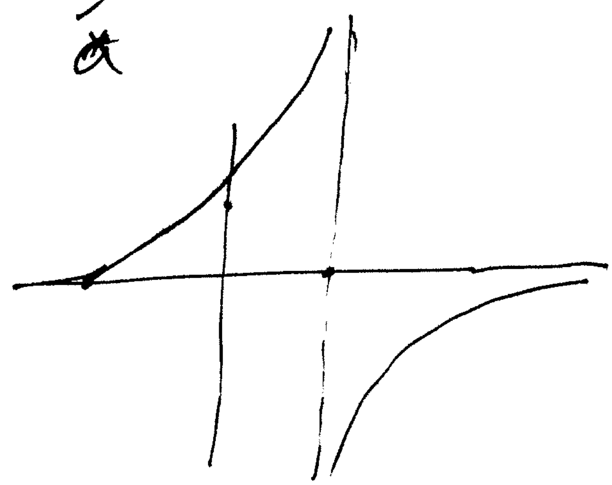
$$\Rightarrow 2\sin^2\theta = 1 - \cos 2\theta$$

$$\Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\Rightarrow 4\sin^2\theta \, d\theta = [2 - 2\cos 2\theta]\, d\theta .$$

Hence

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx = \int_{a}^{b} [2 - 2\cos 2\theta]\, d\theta$$

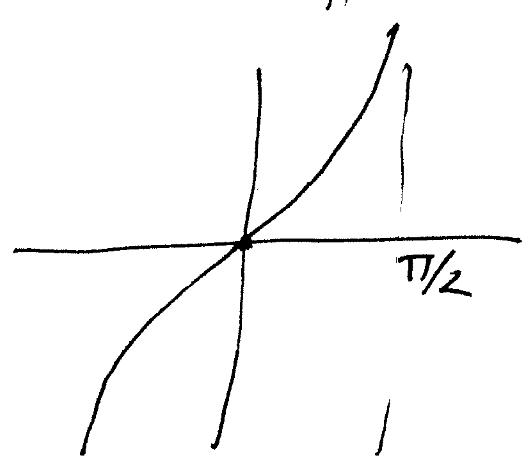


when $x = -1$ $\quad z = 0$

when $x = 1$ $\quad z = \infty$

when $z = 0$ $\quad \theta = 0$

when $z = \infty$ $\quad \theta = \pi/2$



Hence $a = 0$ $b = \pi/2$

and we have

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx = \int_{0}^{\pi/2} (2 - 2\cos 2\theta)\, d\theta$$

$$= 2\theta - \cancel{2}\frac{\sin 2\theta}{\cancel{2}} \Big|_{0}^{\pi/2}$$

$$= \left(2 \cdot \frac{\pi}{2} - \sin 2\frac{\pi}{2}\right) - \left(2 \cdot 0 - \sin 0\right)$$

$$= \pi$$

$$\therefore \boxed{\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx = \pi}$$

2

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \int_0^{1^-} \frac{dx}{(x-1)^{2/3}} + \int_{1^+}^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \underset{a\to 1}{Lt} \int_0^a \frac{dx}{(x-1)^{2/3}} + \underset{b\to 1}{Lt} \int_b^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \underset{a\to 1}{Lt} \left. \frac{(x-1)^{1/3}}{1/3} \right|_0^a + \underset{b\to 1}{Lt} \left. \frac{(x-1)^{1/3}}{1/3} \right|_b^3$$

$$= \underset{a\to 1}{Lt}\ 3\left[(a-1)^{1/3} - (-1)^{1/3}\right] + \underset{b\to 1}{Lt}\ 3\left[(3-1)^{1/3} - (b-1)^{1/3}\right].$$

$$= 3\left[0+1\right] + 3\left[2^{1/3} - 0\right]$$

$$= 3\left[1 + \sqrt[3]{2}\right]$$

3 $$\int_1^{\infty} \frac{dx}{x^2} = \underset{a\to\infty}{Lt} \int_1^{a} \frac{dx}{x^2}$$

$$\int \frac{dx}{x^2} = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x}$$

$$\int_1^a \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^a = -\frac{1}{a} + 1$$

$$\underset{a\to\infty}{Lt} \int_1^a \frac{dx}{x^2} = \underset{a\to\infty}{Lt}\left[1 - \frac{1}{a}\right] = 1$$

4 In the interval $[1, \infty)$ we have

$$e^{-x^2} \leq e^{-x}$$

Hence

$$\int_1^{a} e^{-x^2} dx \leq \int_1^a e^{-x} dx$$

$$= -e^{-x}\Big|_1^a = -e^{-a} + e^{-1}$$

Thus.

$$\int_1^a e^{-x^2}dx \leq \frac{1}{e} - \frac{1}{e^a} < \frac{1}{e}$$

for $a \geq 1$

Hence

$$\int_1^\infty e^{-x^2}dx < \frac{1}{e}$$

It would follow that this limit exists.

5 $\int \cos x\,dx = \sin x$

$$\Rightarrow \int_0^a \cos x\,dx = \sin a - \sin 0 = \sin a.$$

$$\Rightarrow \int_0^\infty \cos x\,dx = \operatorname*{Lt}_{a\to\infty} \sin a$$

which does not exist.