H. W. 7 (Answers)

1 Ans:

(i) $I = \int \frac{x^2+3x+9}{(x^2+2x+2)^2}\,dx$

We write

$$\frac{x^2+3x+9}{(x^2+2x+2)^2} = \cancel{\frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{(x^2+2x+2)^2}}$$

write

$$x^2+2x+2 = (x+1)^2 + 1$$

$$(x^2+3x+9) = (x+1)^2 + x + 8$$

$$= (x+1)^2 + (x+1) + 7.$$

Let $z = x+1$, $dz = dx$ and we have

$$I = \int \frac{z^2+z+7}{(z^2+1)^2}\,dz$$

we write

$$\frac{z^2+z+7}{(z^2+1)^2} = \frac{Az+B}{z^2+1} + \frac{Cz+D}{(z^2+1)^2}$$

$$= \frac{Az}{z^2+1} + \frac{B}{z^2+1} + \frac{Cz+D}{(z^2+1)^2}$$

$$\int \frac{Az}{z^2+1}\,dz = \int \frac{A\,dw}{2w} \qquad \begin{cases} w = z^2+1 \\ dw = 2z\,dz \end{cases}$$

$$= \frac{A}{2}\ln|w| = \frac{A}{2}\ln(z^2+1)$$

$$= \frac{A}{2}\ln(x^2+2x+2)$$

$$\int \frac{B}{z^2+1}\,dz = B\tan^{-1} z$$

$$= B\tan^{-1}(x+1)$$

$$\frac{Cz+D}{(z^2+1)^2} = \frac{Cz}{(z^2+1)^2} + \frac{D}{(z^2+1)^2}$$

$$\int \frac{Cz}{(z^2+1)^2}\, d\cancel{x}z$$

$w = z^2+1$

$dw = 2z\, dz$

$$= \frac{C}{2} \frac{dw}{w^2}$$

$$= \frac{C}{2}\left(-w^{-1}\right) = -\frac{C}{2}\frac{1}{w}$$

$$= -\frac{C}{2}\frac{1}{z^2+1}$$

$$= -\frac{C}{2}\frac{1}{x^2+2x+2}$$

$$\int \frac{D}{(z^2+1)^2}\, dz \qquad\qquad z = \tan\theta$$

$$dz = \sec^2\theta\, d\theta$$

$$z^2+1 = 1+\tan^2\theta$$
$$= \sec^2\theta$$

$$= D\int \frac{\sec^2\theta\, d\theta}{\sec^4\theta}$$

$$= D\int \cos^2\theta\, d\theta$$

$$= \frac{D}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{D}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$$

$$= \frac{D}{2}\tan^{-1} z + \frac{D}{4} 2\sin\theta\cos\theta$$

$$= \frac{D}{2}\tan^{-1}(x+1) + \frac{D}{2}\sin\theta\cos\theta$$

$$= \frac{D}{2}\tan^{-1}(x+1) + \frac{D}{2}\,\frac{x+1}{x^2+2x+2}.$$

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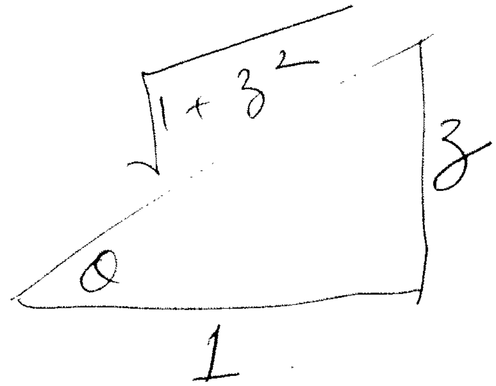

$$\sin\theta = \frac{z}{\sqrt{1+z^2}}, \quad \cos\theta = \frac{1}{\sqrt{1+z^2}}.$$

$$\sin\theta\cos\theta = \frac{z}{1+z^2}.$$

$$= \frac{x+1}{x^2+2x+2}.$$

Collecting all the terms, we have

$$I = \frac{A}{2}\ln(x^2+2x+2)$$

$$+ \; B\tan^{-1}(x+1).$$

$$- \; \frac{C}{2} \; \frac{1}{x^2+2x+2}.$$

$$+ \; \frac{D}{2}\tan^{-1}(x+1)$$

$$+ \; \frac{D}{2} \; \frac{x+1}{x^2+2x+2}$$

Finally, we proceed to calculate A, B, C, D as follows.

$$\frac{z^2+z+7}{(z^2+1)^2} = \frac{(Az+B)(z^2+1)+Cz+D}{(z^2+1)^2}$$

$$= \frac{Az^3+Az+Bz^2+B+Cz+D}{(z^2+1)^2}$$

$$= \frac{Az^3+Bz^2+(A+C)z+(B+D)}{(z^2+1)^2}$$

We have $A=0$, $B=1$

$A+C=1$, $B+D=7$.

$D=6$, $C=1$

Hence

$$I = \tan^{-1}(x+1)$$

$$- \frac{1}{2} \frac{1}{x^2+2x+2}$$

$$+ 3\tan^{-1}(x+1)$$

$$+ 3 \frac{x+1}{x^2+2x+2}$$

$$= 4\tan^{-1}(x+1) + \frac{-1+6(x+1)}{2(x^2+2x+2)}$$

$$I = 4\tan^{-1}(x+1) + \frac{6x+5}{2(x^2+2x+2)}$$

1 ii

$$\int \frac{x^2+3x+9}{(x+1)^2(x+2)^2}\,dx$$

We write

$$\frac{x^2+3x+9}{(x+1)^2(x+2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} \qquad (*)$$

$$\int \frac{A}{x+1}\,dx = A\ln|x+1|$$

$$\int \frac{C}{x+2}\,dx = C\ln|x+2|.$$

$$\int \frac{B}{(x+1)^2}\,dx = \qquad y = x+1$$
$$dy = dx$$

$$\int \frac{B}{y^2}\,dy$$

$$= -B/y = -\frac{B}{x+1}$$

$$\int \frac{D}{(x+2)^2}\,dx = -\frac{D}{x+2}$$

collecting all the terms we have

$$I = A\ln(x+1) + C\ln(x+2)$$
$$-\frac{B}{x+1} - \frac{D}{x+2}.$$

The constants A, B, C, D are calculated as follows:

Multiply * by $(x+1)^2$ and set $x=-1$ to get

$$\frac{1-3+9}{1^2} = B \Rightarrow B = 7$$

Multiply * by $(x+2)^2$ and set $x=-2$ to get

$$\frac{4-6+9}{1^2} = D \Rightarrow D = 7$$

The right hand side of ⊛ is given by

$$\frac{A}{x+1} + \frac{7}{(x+1)^2} + \frac{C}{x+2} + \frac{7}{(x+2)^2}$$

$$= \frac{A(x+1)(x+2)^2 + 7(x+2)^2 + C(x+2)(x+1)^2 + 7(x+1)^2}{(x+1)^2(x+2)^2}$$

The numerator equals

$$A(x+1)(x^2+4x+4) + 7(x^2+4x+4) + C(x+2)(x^2+2x+1) + 7(x^2+2x+1)$$

$$= A(x^3+5x^2+8x+4) + 7(x^2+4x+4) + C(x^3+4x^2+5x+2) + 7(x^2+2x+1)$$

$$= (A+C)x^3$$

$$+ (5A + 7 + 4C + 7)x^2$$

$$+ \text{other terms}$$

$$= x^2 + 3x + 9$$

$$A + C = 0,$$

$$5A + 4C + 14 = 1$$

$$\Downarrow \quad A = -C$$

$$A = -13, \quad C = 13$$

$I =$

$$-13 \ln(x+1) + 13 \ln(x+2) - 7 \underbrace{\left[\frac{1}{x+1} + \frac{1}{x+2} \right]}_{\frac{2x+3}{(x+1)(x+2)}}$$

$$I = 13 \ln(x+2) - 13 \ln(x+1) - \frac{14x+21}{(x+1)(x+2)}.$$

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② Ans:

(i) $\frac{dy}{dx} + \sec x \, y = \sin 2x$

$P(x) = \sec x$, $Q(x) = \sin 2x$

$I(x) = e^{\int \sec x \, dx}$

$= e^{\ln(\sec x + \tan x)}$

$I(x) = \sec x + \tan x$

$y(x) = \frac{1}{\sec x + \tan x}\left[\int (\sec x + \tan x)[\sin 2x]\, dx + C\right]$

$[\sec x + \tan x] \sin 2x$

$= \left[\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right] 2 \sin x \cos x$

$= \frac{(1+\sin x)}{\cancel{\cos x}} \, 2 \sin x \cancel{\cos x} = 2 \sin x \, (1 + \sin x)$

cos 2

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$$\int \left[2\sin x + 2\sin^2 x\right] dx$$

$$= \int \left[2\sin x + 1 - \cos 2x\right] dx$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$2\sin^2 x = 1 - \cos 2x$$

$$= -2\cos x + x - \frac{\sin 2x}{2}$$

$$\therefore y(x) = \frac{1}{\sec x + \tan x}\left[x - 2\cos x - \frac{1}{2}\sin 2x + C\right]$$

2 ii

$$\frac{dy}{dx} + \frac{1}{x}y = \tan^{-1}x$$

$$P(x) = \frac{1}{x} \; ; \; Q(x) = \tan^{-1}x$$

$$I(x) = e^{\int \frac{1}{x} dx} = x$$

$$y(x) = \frac{1}{x}\left[\int x\tan^{-1}x \, dx + C\right]$$

$$\tan^{-1}x = y$$

$$x = \tan y$$

$$dx = \sec^2 y \, dy$$

$$\int x\tan^{-1}x \, dx = \int y\tan y \sec^2 y \, dy$$

$$= \int y \, \frac{\sin y}{\cos^3 y} \, dy .$$

$$\int \frac{\sin y}{\cos^3 y} \, dy \qquad \begin{aligned} w &= \cos y \\ dw &= -\sin y \, dy \end{aligned}$$

$$= \int \frac{-dw}{w^3}$$

$$= (-1) \frac{w^{-2}}{-2} = \frac{1}{2} \frac{1}{w^2}$$

$$= \frac{1}{2} \frac{1}{\cos^2 y} .$$

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$$\int y \frac{\sin y}{\cos^3 y}\, dy$$

$$= y\left(\frac{1}{2}\frac{1}{\cos^2 y}\right) -$$

$$\int \frac{1}{2}\frac{1}{\cos^2 y}\, dy.$$

$$= \frac{y}{2}\sec^2 y - \frac{1}{2}\tan y.$$

$$= \frac{\tan^{-1} x}{2}\sec^2\left[\tan^{-1}(x)\right] - \frac{1}{2}x$$

$$= \frac{1}{2}\tan^{-1} x\left[1+x^2\right] - \frac{x}{2}.$$

$$Y(x) = \frac{1}{x}\left[\frac{1+x^2}{2}\tan^{-1} x - \frac{x}{2} + c\right]$$

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$$\frac{dy}{dx} + \tan x \, y = \sin x .$$

$$P(x) = \tan x \ ; \ Q(x) = \sin x$$

$$I(x) = e^{\int \tan x \, dx}$$

$$\int \tan x = -\ln|\cos x|$$

$$I(x) = \frac{1}{|\cos x|}$$

$$y(x) = |\cos x| \left[\int \frac{\sin x \, dx}{|\cos x|} + C \right]$$

$$\cancel{= |\cos x|\left[-\ln(\cos x) + C\right]}$$

~~when $\cos x$~~

$\cancel{w = \cos x}$

$\cancel{dw = -\sin x\,dx}$

$$= -\cos x\left[\ln|\cos x| + C\right]$$

when $\cos x > 0$

$|\cos x| = \cos x$

$\dfrac{\sin x}{|\cos x|} = \tan x$

$\cos x \int \tan x \, dx$

$= -\cos x \ln|\cos x|$

when $\cos x < 0$

$|\cos x| = -\cos x$

$\dfrac{\sin x}{|\cos x|} = -\tan x$

$-\cos x \int -\tan x \, dx$

$= -\cos x \ln|\cos x|$

③

(i) $$\int \frac{1}{x^3}\,dx = \frac{x^{-2}}{-2} = -\frac{1}{2}\,\frac{1}{x^2}$$

$$\int_1^N \frac{1}{x^3}\,dx = -\frac{1}{2}\,\frac{1}{x^2}\Big|_1^N$$

$$= -\frac{1}{2}\left[\frac{1}{N^2} - 1\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{N^2}\right]$$

$$\int_1^\infty \frac{1}{x^3}\,dx = \operatorname{Lt}_{N\to\infty} \frac{1}{2}\left[1 - \frac{1}{N^2}\right]$$

$$= \frac{1}{2}$$

3 ii $\int x^{-1/2}\,dx = \dfrac{x^{1/2}}{1/2} = 2x^{1/2}$

$$\int_1^N x^{-1/2}\,dx = 2\left[N^{1/2} - 1\right]$$

$$\int_1^\infty x^{-1/2}\,dx = \underset{N\to\infty}{Lt}\ 2\left[N^{1/2} - 1\right]$$

diverges.

3 iii $\displaystyle\int_3^\infty \frac{dx}{2x-1}$

$$\int_3^N \frac{dx}{2x-1} = \frac{1}{2}\ln(2x-1)\Big|_3^N$$

$$= \frac{1}{2}\left[\ln(2N-1) - \ln(5)\right]$$

$\underset{N\to\infty}{Lt} \displaystyle\int_3^N \frac{dx}{2x-1}$ diverges.

(iv)

$$\int 5e^{-2x}\,dx = \frac{5e^{-2x}}{-2}$$

$$= -\frac{5}{2}e^{-2x}$$

$$\int_0^N 5e^{-2x}\,dx = -\frac{5}{2}\left[e^{-2N} - 1\right]$$

$$\underset{N\to\infty}{Lt}\int_0^N 5e^{-2x}\,dx = -\frac{5}{2}(-1)$$

$$= 5/2$$

(v)

$$\int \ln x\,dx = x\ln x - x$$

$$\int_1^N \ln x\,dx = (N\ln N - N) - (1\cancel{\ln 1}^{0} - 1)$$

$$= 1 + N(\ln N - 1).$$

$$\int_1^\infty \ln x\,dx \Rightarrow \text{diverges}.$$