Review Questions for the Final

①

ⓐ Calculate the two l.i. eigenvectors of the matrix

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

Ans: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

after calculating the eigenvalues.

ⓑ Find a set of 3 l.i. ~~all~~ eigenvectors or generalized eigenvectors of ~~the~~ matrix

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Ans: $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

② Showing all steps, calculate

$$e^{\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} t}$$

by hand.

Ans $\begin{pmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{pmatrix}$

③ Solve the following ode

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -x_2 + 1 \qquad x_1(0)=0,\ x_2(0)=0$$

to calculate $x_1(t)$

$$\boxed{\text{Ans: } 1 - e^{-t} - te^{-t}}$$

④ Consider a linear transformation

$$T: \mathbb{R}^4 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \\ z \\ \omega \end{pmatrix} \longmapsto \begin{pmatrix} x+3y+7z+\omega \\ 5x+2y+6z+\omega \end{pmatrix}$$

ⓐ Calculate a basis of the null space of T.

ⓑ Find all points p in $\mathbb{R}^4$: $T(p) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Ans: ⓐ $b_1 = \begin{pmatrix} 1 \\ 0 \\ 4 \\ -29 \end{pmatrix}$; $b_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 4 \end{pmatrix}$ is one choice of basis.

Ans (b) :

$$p = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 9 \end{pmatrix} \text{ is in } T^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

$$\therefore T^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 9 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 0 \\ 4 \\ -29 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 4 \end{pmatrix} \right\}$$

⑤ Consider the following discrete iteration:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1/8 & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = x_1(k)$$

Assume $x_1(0) = x_2(0) = 0$ and $u(k) = 1 \;\forall k$.
Using Jury's test, verify when $y(k)$ remains bounded for every value of k.

Hint: Need to verify that the roots of the polynomial $\lambda^2 + \frac{1}{4}\lambda - \frac{1}{8}$ have magnitude < 1.

⑥ Project the vector $\begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix}$ onto the plane spanned by

$$\begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

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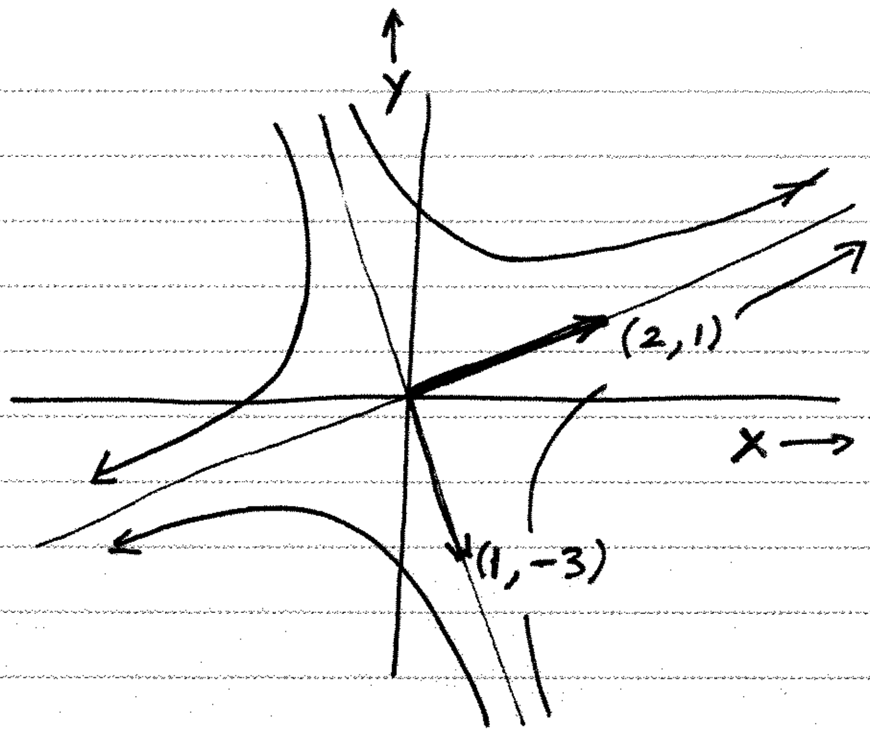

Find example of one dynamical system that has the above phase portrait.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$