Syllabus for Midterm II

1. Row space, Column space and Null space computations of a matrix. Computing basis vectors of these spaces.

2. Projection of a vector on another vector or a plane. Gram-Schmidt orthogonalization.

3. • Eigenvalues, Eigenvectors, Generalized Eigenvectors.
   • Diagonalization, Jordanization by similarity transformation.
   • Calculating e^{At}.

Some problems to ponder :—

1. The matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -16 & -7 \end{pmatrix}$$

has eigenvalues at $\lambda = -2, -2, -3$.

- For the eigenvalue $\lambda = -2$, calculate one eigenvector and one generalized eigenvector. For the eigenvalue $\lambda = -3$, calculate one eigenvector.
- We want to write
$$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$
How to calculate $\alpha_0, \alpha_1, \alpha_2$.
- We want to write
$$A^{98} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$
How to calculate $\alpha_0, \alpha_1, \alpha_2$

2.
$$M = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 7 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 5 & 10 \end{pmatrix}$$

- Obtain cartesian equations of the row space, column space and null space of M.
- Obtain a basis of each of these spaces.

3. • Project the vector

$$u = (1 \quad 2 \quad -1 \quad 3 \quad 9)$$

onto the plane P spanned by the two vectors.

$$v_1 = (0 \quad 1 \quad -1 \quad 0 \quad 0)$$

and

$$v_2 = (2 \quad 1 \quad 0 \quad 0 \quad 1)$$

- Find co-ordinates of $\text{proj}_P u$ w.r.t $\{v_1, v_2\}$ as basis of P.

1 Ans:

Let $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ be the eigenvector

and

$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ be the gen. eigenvector.

We have

$$AV = -2V \quad , \quad AV = -3V$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -16 & -7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix}, \quad \lambda = -2, -3.$$

$$\Rightarrow v_2 = \lambda v_1, \; v_3 = \lambda v_2 = \lambda^2 v_1 .$$

$$-12 v_1 - 16 \lambda v_1 - 7 \lambda^2 v_1 = \lambda^3 v_1$$

$$\Rightarrow \boxed{\lambda^2 + 7\lambda^2 + 16\lambda + 12 = 0}$$

λ satisfies char. poly.

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For $\lambda = -2$

$$v = \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

$\lambda = -3$

$$v = \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix}$$

— x —

$$Au = -2u + v$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -16 & -7 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -2u_1 \\ -2u_2 \\ -2u_3 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

$$u_2 = -2u_1 + 1$$

$$u_3 = -2u_2 - 2 = -2(-2u_1 + 1) - 2$$

$$= 4u_1 - 4$$

$$-12u_1 - 16u_2 - 7u_3 = -2u_3 + 4$$

$$\Rightarrow 5u_3 + 16u_2 + 12u_1 + 4 = 0$$

$$\Rightarrow 20u_1 - 20 - 32u_1 + 16 + 12u_1 + 4 = 0$$

which is always satisfied.

③

Hence

$$u = \begin{pmatrix} u_1 \\ -2u_1+1 \\ 4u_1-4 \end{pmatrix}$$ is a gen. eigenvector for any value of u_1.

choose $u_1 = 0$ $\quad u = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$

—— × ——

$$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

To solve for $\alpha_0, \alpha_1, \alpha_2$ we write

$$e^{\lambda t} = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2$$

$$te^{\lambda t} = 0 + \alpha_1 + 2\alpha_2 \lambda \leftarrow \text{Since } \lambda = -2 \text{ repeats}$$

—— × ——

$$e^{-2t} = \alpha_0 - 2\alpha_1 + 4\alpha_2 \leftarrow \lambda = -2$$

$$te^{-2t} = \alpha_1 - 4\alpha_2 \leftarrow \lambda = -2$$

$$e^{-3t} = \alpha_0 - 3\alpha_1 + 9\alpha_2 \leftarrow \lambda = -3.$$

④

$$\begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 1 & -3 & 9 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} e^{-2t} \\ te^{-2t} \\ e^{-3t} \end{pmatrix}$$

— x ——.

$$A^{98} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2.$$

$$\lambda^{98} = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2.$$

$$98\lambda^{97} = 0 + \alpha_1 + 2\alpha_2 \lambda.$$

$$\begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 1 & -3 & 9 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} (-2)^{98} \\ 98(-2)^{97} \\ (-3)^{98} \end{pmatrix}$$

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2 $$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 7 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 5 & 10 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 2 & 5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3\cdot5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -9\cdot5 \\ 0 & 0 & 1 & 3\cdot5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Row space is spanned by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -\frac{19}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 7/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} \leftarrow \text{Dim} = 3.$$

Null space is described by

$$\left.\begin{aligned} x_1 - 9.5\, x_4 &= 0 \\ x_3 + 3.5\, x_4 &= 0 \\ x_2 + 2x_4 &= 0 \end{aligned}\right\} \quad \boxed{\begin{aligned} 2x_1 - 19x_4 &= 0 \\ 2x_3 + 7x_4 &= 0 \\ x_2 + 2x_4 &= 0 \end{aligned}}$$

$$x_1 = \frac{19}{2} x_4 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 19/2 \\ -2 \\ -7/2 \\ 1 \end{pmatrix} x_4$$

$$x_2 = -2x_4$$

$$x_3 = -\frac{7}{2} x_4$$

Null space is spanned by $\begin{pmatrix} 19 \\ -4 \\ -7 \\ 2 \end{pmatrix} \leftarrow \text{Dim } 1.$

Eqn of the row space

$$19x_1 - 4x_2 - 7x_3 + 2x_4 = 0$$

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