Answers to H. W. 4

1. Let $(x_1, x_2, x_3, x_4, x_5)$ be the set of vectors in S. We have

$$x_1 - x_3 + 2x_4 - 3x_5 = 0$$
$$2x_1 + x_2 - x_5 = 0$$

we can describe S as

$$S = \left\{ \begin{pmatrix} x_3 - 2x_4 + 3x_5 \\ -2x_3 + 4x_4 - 5x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} : x_3, x_4, x_5 \in \mathbb{R} \right\}$$

$$//$$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -2 \\ 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} 3 \\ -5 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_5$$

Define

$$V_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} -2 \\ 4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 3 \\ -5 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

S is spanned by the set of three independent vectors V_1, V_2, V_3.

a. Basis of $S = \{V_1, V_2, V_3\}$

b. $\dim S = 3$

c. Gram-Schmidt orthogonalization:

$$u_1 = V_1 = (1 \quad -2 \quad 1 \quad 0 \quad 0)$$

$$u_2 = \left(V_2 - \frac{V_2 \cdot u_1}{u_1 \cdot u_1} u_1\right) \times 3 = (-1 \quad 2 \quad 5 \quad 3 \quad 0)$$

This multiplication ensures that u_2 has all integers in its components. This is allowed because scaling does not change orthogonality.

$$u_3 = \left(v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 \right) \times 2$$

$$= \begin{pmatrix} 1 & 0 & -1 & 2 & 2 \end{pmatrix}$$

$\{u_1, u_2, u_3\}$ is an orthogonal set of vectors that form a basis of S.

d. Let $v = \begin{pmatrix} 3 & 1 & -1 & 2 & -4 \end{pmatrix}$

<u>Method 1</u>

$$\text{proj}_S v = \text{proj}_{[u_1 \ u_2 \ u_3]} v$$

$$= \text{proj}_{u_1} v + \text{proj}_{u_2} v + \text{proj}_{u_3} v$$

This is true only when u_1, u_2, u_3 are pairwise orthogonal.

It turns out that each of the three projections equal zero. Hence

$\text{proj}_S v = 0.$

Method 2

Let $\omega = \text{proj}_S V$, it follows that

$$(V - \omega) \perp S$$

Since $\{V_1, V_2, V_3\}$ span S we have

$$\omega = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. Also we have

$$(V - \omega) \perp S$$

$$\Rightarrow (V-\omega) \cdot V_1 = 0$$
$$(V-\omega) \cdot V_2 = 0$$
$$(V-\omega) \cdot V_3 = 0$$

$$\Rightarrow \omega \cdot V_1 = V \cdot V_1$$
$$\omega \cdot V_2 = V \cdot V_2$$
$$\omega \cdot V_3 = V \cdot V_3$$

$$\Rightarrow \alpha_1 (V_1 \cdot V_1) + \alpha_2 (V_2 \cdot V_1) + \alpha_3 (V_3 \cdot V_1) = (V \cdot V_1)$$
$$\alpha_1 (V_1 \cdot V_2) + \alpha_2 (V_2 \cdot V_2) + \alpha_3 (V_3 \cdot V_2) = (V \cdot V_2)$$
$$\alpha_1 (V_1 \cdot V_3) + \alpha_2 (V_2 \cdot V_3) + \alpha_3 (V_3 \cdot V_3) = (V \cdot V_3)$$

Since $V \cdot V_1 = V \cdot V_2 = V \cdot V_3 = 0$ we have

$\alpha_1 = \alpha_2 = \alpha_3 = 0 \Rightarrow \omega = 0$

2. Writing

$$\begin{pmatrix} t_1 \\ t_1+t_2 \\ t_1+t_2+t_3 \\ t_2+t_3 \\ t_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1\\1\\1\\0\\0 \end{pmatrix}}_{V_1} t_1 + \underbrace{\begin{pmatrix} 0\\1\\1\\1\\0 \end{pmatrix}}_{V_2} t_2 + \underbrace{\begin{pmatrix} 0\\0\\1\\1\\1 \end{pmatrix}}_{V_3} t_3$$

(a) Basis $= \{v_1, v_2, v_3\}$, $\dim S = 3$

(b) $(1 \quad 0 \quad 0 \quad 0 \quad 0)$ does not belong to S.

Hence co-ordinates cannot be calculated.

(c) $u_1 = v_1 = (1 \quad 1 \quad 1 \quad 0 \quad 0)$

$$u_2 = \left(v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1\right) \times 3 = (-2 \quad 1 \quad 1 \quad 3 \quad 0)$$

$$u_3 = \left(v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2\right) \times 5$$

$$= (1 \quad -3 \quad 2 \quad 1 \quad 5)$$

(d) yes, choose $t_1 = 1, t_3 = 1, t_2 = 0$

(e) $v_1 \cdot v_1 = 3, \quad v_2 \cdot v_1 = 2, \quad v_3 \cdot v_1 = 1$

$v_1 \cdot v_2 = 2, \quad v_2 \cdot v_2 = 3, \quad v_3 \cdot v_2 = 2$

$v_1 \cdot v_3 = 1, \quad v_2 \cdot v_3 = 2, \quad v_3 \cdot v_3 = 3$

Let $u = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\quad u \cdot v_1 = 3$, $\quad u \cdot v_2 = 3$, $\quad u \cdot v_3 = 3$

Define $\operatorname{proj}_S u = w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

We have from page 4, the following.

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

$\leftarrow$ Need to solve this.

2) a) The set of vectors in S can be written as

$$\underbrace{\begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix}}_{v_1} t_1 + \underbrace{\begin{pmatrix}0\\1\\1\\1\\0\end{pmatrix}}_{v_2} t_2 + \underbrace{\begin{pmatrix}0\\0\\1\\1\\1\end{pmatrix}}_{v_3} t_3 , \quad t_1, t_2, t_3 \in \mathbb{R}$$

Hence $S = \text{span}[v_1, v_2, v_3]$

To show that v_1, v_2, v_3 are independent we write

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\Rightarrow \left.\begin{array}{ll} \alpha_1 = 0 & \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + \alpha_2 = 0 & \alpha_3 = 0 \\ \alpha_1 + \alpha_2 + \alpha_3 = 0 & \end{array}\right\} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Hence

$$\{v_1, v_2, v_3\}$$

forms a basis of S.

$$\text{Dim } S = 3.$$

(b) Let $u = (1\ 0\ 0\ 0\ 0)$. We write

$$u = a v_1 + b v_2 + c v_3$$

$$\left.\begin{array}{l} a = 1 \\ b + a = 0 \\ c + b + a = 0 \\ b + c = 0 \\ c = 0 \end{array}\right\} \Rightarrow a = 1,\ b = -1,\ c = 0$$

$b + c$ cannot be 0.

Hence $u \notin S$. We cannot calculate the co-ordinates.

c

9

```
  Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

  To get started, select "MATLAB Help" from the Help menu.

>> x1=[1 1 1 0 0];
>> x2=[0 1 1 1 0];
>> x3=[0 0 1 1 1];
>> p1=x1/(sqrt(dot(x1,x1)))

p1 =

    0.5774    0.5774    0.5774         0         0

>> v2=x2-dot(p1,x2)*p1

v2 =

   -0.6667    0.3333    0.3333    1.0000         0

>> p2=v2/sqrt(dot(v2,v2))

p2 =

   -0.5164    0.2582    0.2582    0.7746         0

>> v3=x3-dot(p1,x3)*p1-dot(p2,x3)*p2

v3 =

    0.2000   -0.6000    0.4000    0.2000    1.0000

>> p3=v3/sqrt(dot(v3,v3))

p3 =

    0.1581   -0.4743    0.3162    0.1581    0.7906

>> [dot(p1,p2) dot(p1,p3)  dot(p2,p3)]

ans =

  1.0e-015 *

   -0.2220    0.0833    0.0694

>>
```

p_1, p_2, p_3 are the required orthonormal vectors.

(d) $u = (1 \quad 1 \quad 2 \quad 1 \quad 1)$

If $u \in S$, it would follow that

$t_1 = 1,\ t_1 + t_2 = 1,\ t_1 + t_2 + t_3 = 2$

$t_2 + t_3 = 1,\ t_3 = 1$

$\Downarrow$

$\boxed{t_1 = 1,\ t_2 = 0,\ t_3 = 1}$

Hence $u \in S$.

(e) $proj_S u = (u \cdot p_1) p_1 + (u \cdot p_2) p_2 + (u \cdot p_3) p_3$

where p_1, p_2, p_3 are the three orthonormal vectors in part (c)

```
u=[1   1   1   1   1];
proj=dot(u,p1) * p1+dot(u,p2) * p2+dot(u,p3) * p3


proj =

(0.7500   0.7500   1.5000   0.7500   0.7500)
```

$$\text{proj}_S\, u = \left(\frac{3}{4}, \frac{3}{4}, \frac{3}{2}, \frac{3}{4}, \frac{3}{4}\right)$$

(f) Let θ be the angle

$$\cos\theta = \frac{u \cdot \text{proj}}{\|u\|\ \|\text{proj}\|} = .9487$$

$\theta = 18.432$ degrees

3

12

```
Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

To get started, select "MATLAB Help" from the Help menu.

>> A=[3 2 1 -1;5 4 5 0;6 -7 6 -5;9 -8 -12 4]

A =

     3     2     1    -1
     5     4     5     0
     6    -7     6    -5
     9    -8   -12     4

>> M11=det([A(2,2) A(2,3) A(2,4);A(3,2) A(3,3) A(3,4);A(4,2) A(4,3) A(4,4)])

M11 =

   196

>> M12=det([A(2,1) A(2,3) A(2,4);A(3,1) A(3,3) A(3,4);A(4,1) A(4,3) A(4,4)])

M12 =

  -525

>> M13=det([A(2,1) A(2,2) A(2,4);A(3,1) A(3,2) A(3,4);A(4,1) A(4,2) A(4,4)])

M13 =

  -616

>> M14=det([A(2,1) A(2,2) A(2,3);A(3,1) A(3,2) A(3,3);A(4,1) A(4,2) A(4,3)])

M14 =

        1239

>> M21=det([A(1,2) A(1,3) A(1,4);A(3,2) A(3,3) A(3,4);A(4,2) A(4,3) A(4,4)])

M21 =

  -136

>> M22=det([A(1,1) A(1,3) A(1,4);A(3,1) A(3,3) A(3,4);A(4,1) A(4,3) A(4,4)])

M22 =

   -51

>> M23=det([A(1,1) A(1,2) A(1,4);A(3,1) A(3,2) A(3,4);A(4,1) A(4,2) A(4,4)])

M23 =

  -357

>> M24=det([A(1,1) A(1,2) A(1,3);A(3,1) A(3,2) A(3,3);A(4,1) A(4,2) A(4,3)])

M24 =
```

13

```
   663

>> M31=det([A(1,2) A(1,3) A(1,4);A(2,2) A(2,3) A(2,4);A(4,2) A(4,3) A(4,4)])

M31 =

    32

>> M32=det([A(1,1) A(1,3) A(1,4);A(2,1) A(2,3) A(2,4);A(4,1) A(4,3) A(4,4)])

M32 =

   145

>> M33=det([A(1,1) A(1,2) A(1,4);A(2,1) A(2,2) A(2,4);A(4,1) A(4,2) A(4,4)])

M33 =

    84

>> M34=det([A(1,1) A(1,2) A(1,3);A(2,1) A(2,2) A(2,3);A(4,1) A(4,2) A(4,3)])

M34 =

   110

>> M41=det([A(1,2) A(1,3) A(1,4);A(2,2) A(2,3) A(2,4);A(3,2) A(3,3) A(3,4)])

M41 =

   -89

>> M42=det([A(1,1) A(1,3) A(1,4);A(2,1) A(2,3) A(2,4);A(3,1) A(3,3) A(3,4)])

M42 =

   -50

>> M43=det([A(1,1) A(1,2) A(1,4);A(2,1) A(2,2) A(2,4);A(3,1) A(3,2) A(3,4)])

M43 =

    49

>> M44=det([A(1,1) A(1,2) A(1,3);A(2,1) A(2,2) A(2,3);A(3,1) A(3,2) A(3,3)])

M44 =

   118

>> Cofac=[M11 -M12 M13 -M14;-M21 M22 -M23 M24;M31 -M32 M33 -M34;-M41 M42 -M43 M44]

Cofac =

         196          525         -616        -1239
         136          -51          357          663
          32         -145           84         -110
```

14

```
         89           -50           -49           118

>> Adj=transpose(Cofac)

Adj =

        196           136            32            89
        525           -51          -145           -50
       -616           357            84           -49
      -1239           663          -110           118

>> A*Adj

ans =

       2261             0             0             0
          0          2261             0             0
          0             0          2261             0
          0             0             0          2261

>> Adj*A

ans =

       2261             0             0             0
          0          2261             0             0
          0             0          2261             0
          0             0             0          2261

>> det(A)

ans =

       2261

>> AINV=Adj/det(A)

AINV =

    0.0867    0.0602    0.0142    0.0394
    0.2322   -0.0226   -0.0641   -0.0221
   -0.2724    0.1579    0.0372   -0.0217
   -0.5480    0.2932   -0.0487    0.0522

>> inv(A)

ans =

    0.0867    0.0602    0.0142    0.0394
    0.2322   -0.0226   -0.0641   -0.0221
   -0.2724    0.1579    0.0372   -0.0217
   -0.5480    0.2932   -0.0487    0.0522

>>
```

4 (a) $p(\lambda) =$

$$\det\begin{pmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{pmatrix} = \lambda(\lambda+3)+2 = \lambda^2+3\lambda+2$$

$$= (\lambda+1)(\lambda+2)$$

(b) $$A^2 = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -3 \\ 6 & 7 \end{pmatrix}$$

$$-3A = \begin{pmatrix} 0 & -3 \\ 6 & 9 \end{pmatrix} \; ; \; -2I = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$-3A - 2I = \begin{pmatrix} -2 & -3 \\ 6 & 7 \end{pmatrix}$$

(c) $A^{98} = \alpha_0 I + \alpha_1 A$

To find α_0, α_1 we solve

$$(-1)^{98} = \alpha_0 + \alpha_1(-1)$$
$$(-2)^{98} = \alpha_0 + \alpha_1(-2)$$

$$\Downarrow$$

$$\left.\begin{aligned} 1 &= \alpha_0 - \alpha_1 \\ 2^{98} &= \alpha_0 - 2\alpha_1 \end{aligned}\right\} \quad \begin{aligned} 1 - 2^{98} &= \alpha_1 \\ 2 - 2^{98} &= \alpha_0 \end{aligned}$$

$$A^{98} = \begin{pmatrix} 2-2^{98} & 0 \\ 0 & 2-2^{98} \end{pmatrix} + \begin{pmatrix} 0 & 1-2^{98} \\ -2(1-2^{98}) & -3(1-2^{98}) \end{pmatrix}$$

$$= \begin{pmatrix} 2-2^{98} & 1-2^{98} \\ -2+2^{99} & -1+2^{99} \end{pmatrix}$$

5 $$\begin{pmatrix} 1 & 2 & 1 \\ 2 & -5 & 3 \\ 3 & -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 \\ 20 \\ 10 \end{pmatrix}$$

$$x = \frac{\begin{vmatrix} 13 & 2 & 1 \\ 20 & -5 & 3 \\ 10 & -3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & -5 & 3 \\ 3 & -3 & 1 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} 1 & 2 & 13 \\ 2 & -5 & 20 \\ 3 & -3 & 10 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & -5 & 3 \\ 3 & -3 & 1 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} 1 & 13 & 1 \\ 2 & 20 & 3 \\ 3 & 10 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & -5 & 3 \\ 3 & -3 & 1 \end{vmatrix}}$$

6 (a) $p(\lambda) = (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$

(b) $p(\lambda) = \text{Char of} \begin{pmatrix} 9 & 1 \\ 0 & 9 \end{pmatrix} \cdot \text{Char of} \begin{pmatrix} 8 & 1 \\ 0 & 8 \end{pmatrix}$

$= (\lambda-9)^2 (\lambda-8)^2$

(c) $p(\lambda) = (\lambda-9)^4$

(d) $p(\lambda) = (\lambda-\sigma)^2 + \omega^2$

(e) $p(\lambda) = \text{Char} \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} \cdot \text{Char} \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$

$= \left[(\lambda-\sigma)^2 + \omega^2\right]^2$

(f) $p(\lambda) = \lambda(\lambda^2 + \omega^2)$ where

$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$

g $p(\lambda) = \lambda^3 + 6\lambda^2 + 11\lambda + 6$

h $p(\lambda) = \mathrm{char}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} \cdot \mathrm{char}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix}$

$= \left[\lambda^3 + 6\lambda^2 + 11\lambda + 6\right]^2$

7 $\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ -1 & 1 & 0 & -3 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Null space of A is described by

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} x_1 + x_3 + 2x_4 &= 0 \\ x_2 + x_3 - x_4 &= 0 \end{aligned}}$$

cartesian eqn of the nullspace

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 - 2x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} x_4$$

$$\text{Null space} = \text{Span}\left[\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right]$$

Basis of the null space

Range:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & -1 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Range of A =

$$\text{Span}\left[\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right]$$

Described by

$$a x_1 + b x_2 + c x_3 = 0$$

To find a, b, c we have

$a - c = 0 \Rightarrow a = c$

$b + c = 0 \qquad b = -c.$

$c x_1 - c x_2 + c x_3 = 0$

$\boxed{x_1 - x_2 + x_3 = 0}$ ← cartesian Eqn of the range space.