Local variations and minimal area problem for Carathéodory functions [*]

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Abstract

Two kinds of minimal area problems have been studied. One, with analytic side conditions, was first treated by H. S. Shapiro. Another kind of problem initiated by A. W. Goodman deals with classes of analytic univalent functions with geometric constraints. The minimal area problem for the Carathéodory functions, i.e. analytic functions having positive real part in the unit disk, belongs to this second type. To solve it, we develop a technique in the frame of classical complex analysis that explores symmetrization type geometric transformations and local boundary variations. This method reduces the minimal area problem to a certain boundary value problem for analytic functions. In the case that the latter problem admits an explicit solution the original minimal area problem can be handled. [1]

1 Introduction

Let P denote the Carathéodory class of functions f analytic in the unit disk $\mathbb{U} = \{z : |z| < 1\}$ having positive real part in $\mathbb{U}$ and normalized by $f(0) = 1$, see [13, § 2.5]. For $\alpha \geq 0$, let P_α be the subclass of functions $f \in P$ such that $f'(0) = \alpha$. If f, defined by

$$f(z) = 1 + c_1(f)z + c_2(f)z^2 + \dots \qquad (1.1)$$

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is in P, then $|c_n(f)| \le 2$ by the well known Carathéodory theorem, [13, § 2.5]. In particular, $|c_1(f)| \le 2$ with equality only for the functions

$$p_\theta(z) = \frac{1+e^{i\theta}z}{1-e^{i\theta}z}, \quad \theta \in \mathbb{R}.$$

Thus P_α is nontrivial only for $0 \le \alpha < 2$.

The Dirichlet integral of f

$$D(f) = \int_{\mathbb{U}} |f'|^2 \, d\sigma \tag{1.2}$$

measures the area of the image $f(\mathbb{U})$ counting multiplicity of covering. Since

$$D(f) = \pi \sum_{n=1}^{\infty} n|c_n(f)|^2 \ge \pi\alpha^2 \tag{1.3}$$

for $f \in P_\alpha$, it follows that for $0 \le \alpha \le 1$ the linear function $f_\alpha(z) = 1 + \alpha z$ minimizes $D(f)$ in P_α. When $1 < \alpha < 2$, the polynomial $1 + \alpha z$ is not in P_α any more and the problem of minimizing $D(f)$ in P_α, i.e. the problem of finding

$$A(\alpha) = \min_{f \in P_\alpha} D(f) \tag{1.4}$$

and all functions $f \in P_\alpha$ such that $D(f) = A(\alpha)$, becomes nontrivial. One of the goals of this paper is to present its solution for all $1 < \alpha < 2$.

Theorem 1 *For $1 < \alpha < 2$, let $f \in P_\alpha$. Then*

$$D(f) \ge A(\alpha) = \pi\alpha\tau^{-2}(\alpha - 2(1-\tau)) \tag{1.5}$$

and the unique extremal is a univalent function

$$f_\alpha(z) = \alpha\tau^{-1} \int_{-1}^{z} t^{-1} p_\tau(t)\, dt \tag{1.6}$$

$$= \frac{\alpha}{2\tau^2 z}\Bigg(z^2 - 1 + (1+z)\sqrt{(1-z)^2 + 4\tau z}$$

$$+2(\tau-1)z \log \frac{1 + 2\tau z + z^2 + (1+z)\sqrt{(1-z)^2+4\tau z}}{1-\tau}\Bigg)$$

with the principal branches of the square root and logarithm, where

$$p_\tau(z) = \frac{4\tau z}{(1 - z + \sqrt{(1-z)^2 + 4\tau z})^2} \tag{1.7}$$

is the classical Pick function and τ is the unique solution of the equation

$$\alpha\,[\tau + (1-\tau)\log(1-\tau)] = \tau^2, \qquad 0 < \tau < 1 \tag{1.8}$$

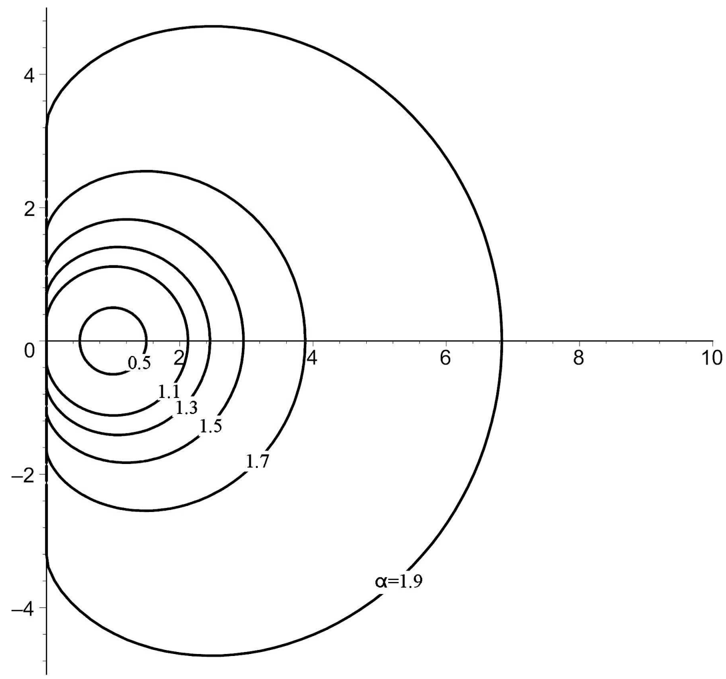

Figure 1: Extremal domains

The Pick function can be represented as $p_\tau(z) = k^{-1}(\tau k(z))$, where $k(z) = z(1-z)^{-2}$ is the Koebe function. p_τ maps $\mathbb{U}$ conformally onto the unit disk $\mathbb{U}$ slit along the segment $[-1, -\tau^{-1}(1-\sqrt{1-\tau})^2]$. Figure 1 shows extremal domains for some values of α, Figure 2 displays a graph of the minimal area $A(\alpha)$.

Corollary 1 *Let $f \in P$ satisfy $c_n(f) = \alpha$ with $0 < \alpha < 2$. Then*

$$D(f) \geq nA(\alpha) \tag{1.9}$$

with the unique extremal

$$f(z) = f_\alpha(z^n). \tag{1.10}$$

.

Of course, inequality (1.9) is nontrivial only for $1 < \alpha < 2$. We give the proofs of Theorem 1 and its corollary in Section 5 after several lemmas.

Although we prefer to speak about the Dirichlet integral, our proof of Theorem 1 actually gives the stronger result showing that (1.5) holds with $D(f)$ replaced by Area $f(U)$ – the area in the w-plane covered by $f(\mathbb{U})$. However, in Corollary 1, the Dirichlet integral, rather than the projected area, is needed for the n-sheeted extremal function. As a consequence of this stronger version of Theorem 1 we get the following variant of Lavrent'ev's inequality for a pair of analytic functions with nonoverlapping images, see [11, § 9].

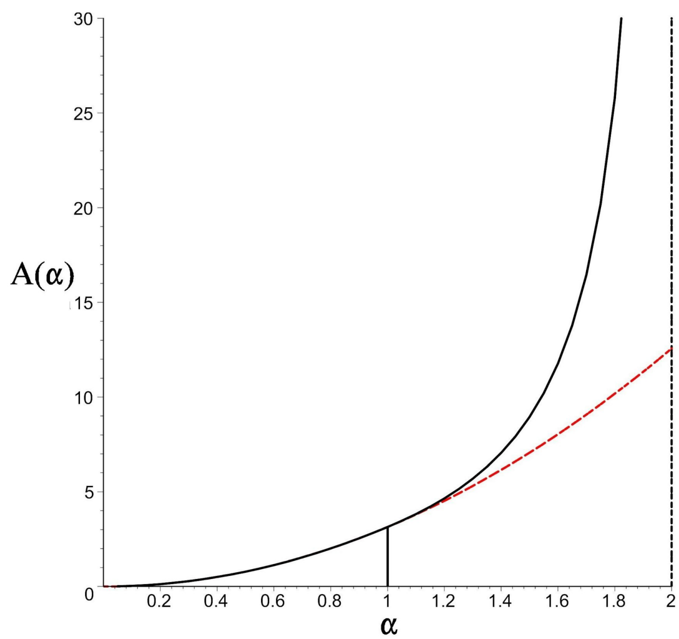

Figure 2: Graph of $A(\alpha)$

Theorem 2 *Let f_1 and f_2 be analytic in $\mathbb{U}$ such that $f_1(\mathbb{U}) \cap f_2(\mathbb{U}) = \emptyset$ and*

$$\text{Area}\, f_1(\mathbb{U}) + \text{Area}\, f_2(\mathbb{U}) = 2S.$$

Then

$$|f_1'(0) f_2'(0)| \le (\alpha^2/4)|f_1(0) - f_2(0)|^2, \tag{1.11}$$

where α is the unique solution of the equation

$$\pi\alpha\tau^{-2}(\alpha)(\alpha - 2(2 - \tau(\alpha)) = 4S|f_1(0) - f_2(0)|^{-2}$$

with $\tau(\alpha)$ defined by (1.8). Equality occurs in (1.11) only for the functions

$$f_1(z) = Af_\alpha(e^{i\theta_1} z) + B, \quad f_2(z) = -Af_\alpha(e^{i\theta_2} z) + B,$$

where $A, B \in \mathbb{C}$, $\theta_1, \theta_2 \in \mathbb{R}$, and f_α is the extremal function of Theorem 1.

Minimal area problems for analytic functions are among the classical subjects of complex analysis. We distinguish two types of them. One is concerned with the minimal area problems with analytic side conditions, e.g. when we fix the value of a function and the values of some of its derivatives at a number of given points. This kind of problem was first studied by H. Shapiro in the 60's. Nontrivial problems of this kind with minimal data are concerned with normalized univalent functions $f(z) = z + a_2(f)z^2 + \ldots$, analytic in $\mathbb{U}$, that have a prescribed second coefficient $a_2(f)$ or a prescribed value at a given point z_0 in $\mathbb{U}$. Solutions to these problems based on the theory of quadrature identities developed by D. Aharonov and Shapiro in [1, 2] and some symmetrization results were found in [3, 4].

The study of another type of minimal area problem, which includes the problem considered in the present paper, with geometric constraints on the image, was initiated by A. W. Goodman [14] and later suggested by D. Brannan (see [6] for a history of the problem). Goodman's omitted area problem for the class S, that is still open, was studied by R. W. Barnard [6], Barnard and K. Pearce [9], and by J. Lewis [19] who used the Alt-Caffarelli variational technique to justify that a boundary condition $|f'| = constant$ is satisfied for extremal functions on the free boundary. A further discussion of Goodman's problem will be given in Section 6.

The method used in [19] involves deep results from PDE's and required a very delicate and lengthy analysis including computation up to some decimals. To avoid these difficulties and the complexities from PDE's our second goal is to develop a more elementary technique, in the frame of classical complex analysis, that allows us to study more general types of minimal area problems. The technique includes three main ingredients. First in Section 2 we apply geometric transformations such as symmetrization, polarization and averaging to study the shape of extremal domains and prove a priori regularity of their boundaries. Our approach gives not only symmetry but also rectifiability and monotonicity of the distortion along certain boundary arcs. For the properties of symmetrization we refer to [16, 17, 3, 5, 11, 23, 24, 26, 27].

A priori properties of extremal domains established in Section 2 allow us to apply elementary boundary variations that is the second ingredient of the method developed in Section 3. Proofs of this section are based on boundary properties of univalent functions and Pommerenke's book [21] is an excellent reference for it.

Results of Sections 2 and 3 lead to a boundary value problem for an extremal analytic function that is the third component of the method. All solved minimal area problems with geometric constraints, which are known to the authors, can be reduced to a mixed boundary value problem for an analytic function with a prescribed real part on a given part of the boundary and a prescribed imaginary part on its complement. The latter problem can be solved by the Keldysh-Sedov formula although in this paper we prefer another more special method. Summing up our knowledge on the extremal functions, we find their closed form in Section 5 that allows us to prove Theorem 1.

In the last section we discuss the present status of Goodman's omitted area problem and a minimal area problem for the hyperbolic metric.

2 Geometry of extremal domains and symmetrization

In this section we describe qualitative properties of extremal domains and the corresponding properties of the extremal functions. Our preference is the usage of geometric methods such as symmetrization and polarization when studying geometric properties of the extremal domains. One advantage of this approach is that it can be easily modified to study minimal area problems with a variety of side conditions.

First we show the existence and uniqueness of the extremal functions. In studying minimal area problems we can restrict ourselves to functions having a finite Dirichlet integral.

Lemma 1 *For every $0 < \alpha < 2$ there is a unique $f \in P_\alpha$ such that $D(f) = A(\alpha)$.*
The minimal area $A(\alpha)$ is a convex, strictly increasing function of α, $0 < \alpha < 2$.

Proof. For a fixed $0 < \alpha < 2$, P_α is a convex set of analytic functions that is compact in the topology of uniform convergence on compact subsets of $\mathbb{U}$. Since the Dirichlet integral is lower semicontinuous the existence of an extremal is immediate.

The Dirichlet integral is convex, i.e.

$$D((f_1 + f_2)/2) = (1/4) \int_{\mathbb{U}} |f_1' + f_2'|^2 \, d\sigma \leq$$

$$(1/2) \left(\int_{\mathbb{U}} |f_1'|^2 \, d\sigma + \int_{\mathbb{U}} |f_2'|^2 \, d\sigma \right) = (1/2)(D(f_1) + D(f_2)) \tag{2.1}$$

with the sign of equality only if $f_2' = c f_1'$ with $c \in \mathbb{C}$. (2.1) implies that for every $0 < \alpha < 2$ the extremal f in P_α is unique.

If $0 < \alpha_1 < \alpha_2 < 1$ and $f \in P_{\alpha_2}$ then $1 + (\alpha_1/\alpha_2)(f - 1)$ is in P_{α_1}. This shows that $\alpha_1^{-2} A(\alpha_1) \leq \alpha_2^{-2} A(\alpha_2)$; in particular $A(\alpha)$ is strictly increasing.

If f_1, f_2 are extremal in P_{α_1} and P_{α_2}, respectively, then $f = (f_1 + f_2)/2$ is in P_α with $\alpha = (\alpha_1 + \alpha_2)/2$. By (2.1),

$$A(\alpha) \leq D(f) \leq (1/2)(D(f_1) + D(f_2)) = (A(\alpha_1) + A(\alpha_2))/2. \tag{2.2}$$

Since $A(\alpha)$ is monotone and satisfies (2.2), it easily follows that $A(\alpha)$ is continuous and therefore convex. $\Box$

Now we recall the definitions of symmetrizations and polarization and introduce the notation of geometric objects that will be used throughout the paper. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{U}$, $\mathbb{T}$, $\mathbb{H}$, and $\mathbb{H}_r$ will be reserved for the real axis, complex plane, unit disk, unit circle, upper half-plane, and right half-plane, respectively. An oriented straight line through z_0 in the direction $e^{i\theta}$ will be denoted by $l_\theta(z_0) = \{z = z_0 + te^{i\theta}, -\infty < t < \infty\}$; $l(x) = l_{\pi/2}(x)$,

$h(y) = l_0(iy)$ with $x, y \in \mathbb{R}$. $[a,b]$ and $]a,b[$ will denote closed and open segments between a and b in $\mathbb{C}$.

The Steiner symmetrization of a domain D w.r.t. $\mathbb{R}$ is a domain D^* such that for every $x \in \mathbb{R}$, $D^* \cap l(x) =]-im, im[$, where $2m = \mu(D \cap l(x))$ and μ stands for the linear Lebesgue measure.

Let $C_r(z_0) = \{z : |z - z_0| = r\}$ with $0 \le r \le \infty$. Thus $C_0(z_0) = z_0$, $C_\infty(z_0) = \infty$. Let $\gamma_\theta(z_0) = \{z = z_0 + te^{i\theta}, 0 \le t < \infty\}$. By circular symmetrization of a domain $D \subset \overline{\mathbb{C}}$ with respect to $\gamma_\theta(z_0)$ we mean, as usual, the domain D^* such that $C_r(z_0) \subset D^*$ if and only if $C_r(z_0) \subset D$ and if $C_r(z_0) \not\subset D$, $0 < r < \infty$ then $D^* \cap C_r(z_0)$ is a proper single arc of $C_r(z_0)$ centered at $z_0 + re^{i\theta}$ such that $\mu\left(D^* \cap C_r(z_0)\right) = \mu\left(D \cap C_r(z_0)\right)$.

A domain D is called starlike with respect to $z_0 \in D$ if D contains $[z_0, z]$ with any point z in D. A domain D^* starlike with respect to z_0 is called the radial symmetrization of D with respect to z_0 if for some $\varepsilon > 0$ such that the disk $\mathbb{U}_\varepsilon(z_0) := \{z : |z - z_0| < \varepsilon\}$ is in D and for all $\theta \in \mathbb{R}$, $\lambda\left((D^* \setminus \mathbb{U}_\varepsilon(z_0)) \cap \gamma_\theta(z_0)\right) = \lambda\left((D \setminus \mathbb{U}_\varepsilon(z_0)) \cap \gamma_\theta(z_0)\right)$, where for any $E \subset \gamma_\theta(z_0)$, $\lambda(E)$ denotes the logarithmic measure of E:

$$\lambda(E) = \int_E |z - z_0|^{-1}\, |dz|.$$

Now we define the *polarization* of a domain $D \subset \overline{\mathbb{C}}$ with respect to $l_\theta(z_0)$ first used by V. Wolontis [28]. Let H^+ and H^- be the left and right half-planes with respect to $l_\theta(z_0)$ and let D^* denote the set symmetric to D with respect to $l_\theta(z_0)$. By the polarization of D with respect to $l_\theta(z_0)$ we mean the set

$$D_p = ((D \cup D^*) \cap \overline{H^+}) \cup ((D \cap D^*) \cap H^-).$$

(See [24] for more details on polarization.) Note that D_p is open but might be disconnected and contain multiply connected components even if D is a simply connected domain.

It is necessary to emphasize that Steiner and circular symmetrizations and polarization preserve the area while the radial symmetrization diminishes it. All of these transformations increase the inner radius of a domain evaluated at appropriate points, [16, 17, 11]. We recall that the inner radius, $R(D, z_0)$, of a domain $D \subset \mathbb{C}$ having Green's function $g(z, z_0)$ with singularity at $z_0 \in D$ is defined by, see [11],

$$\log R(D, z_0) = \lim_{z \to z_0} (g(z, z_0) + \log |z - z_0|).$$

For simply connected domains the inner radius coincides with the conformal radius. In this case the quantity $m(D, z_0) = (1/2\pi) \log R(D, z_0)$ is called *the reduced modulus* of D at z_0, [17].

Lemma 2 *Let f be extremal w.r.t. minimal area in P_α, $1 < \alpha < 2$. Then f maps $\mathbb{U}$ conformally onto a bounded rectifiable Jordan domain $D = f(\mathbb{U})$ that is starlike w.r.t. $w = 1$, possesses Steiner and circular symmetries w.r.t. $\mathbb{R}$ and $\mathbb{R}_1 = \{x \in \mathbb{R} : x \ge 1\}$ respectively, and contains the point $w = 0$ on the boundary.*

Proof. To prove that the extremal domain $D = f(\mathbb{U})$, $f \in P_\alpha$ possesses Steiner symmetry, we assume it does not. Let D^* be the Steiner symmetrization of D w.r.t. $\mathbb{R}$ and let $F(z) = 1 + \beta z + \dots$ with $\beta > 0$ map $\mathbb{U}$ conformally onto D^*. By the principle of symmetrization, see [16, 11], $\beta > \alpha$ with the sign of strict inequality since $D \neq D^*$. For $\tau = \alpha/\beta < 1$, $F_\tau(z) = F(\tau z) \in P_\alpha$ satisfies

$$D(F_\tau) = \alpha^2 + \sum_{n=1}^{\infty} n\tau |a_n(F)|^2 < \beta^2 + \sum_{n=1}^{\infty} n|a_n(F)|^2 = D(F) = \text{Area}\,(D) \le D(f) \quad (2.3)$$

contradicting the extremality of f. Thus D possesses Steiner symmetry and f maps $\mathbb{U}$ conformally and one-to-one onto D.

To prove that D is starlike or possesses circular symmetry we apply radial or circular symmetrizations and follow the same scheme.

Boundedness of D follows from boundedness of f' which we shall prove in Section 4 by applying a variational technique. Here we present another, purely geometric, proof based on polarization. The idea of such a proof was mentioned in [3].

Assume D is not bounded. Then $\mathbb{R}_1 \subset D$ since D is circularly symmetric. Since $\text{Area}\, D < \infty$, for a given $\varepsilon > 0$, there is $u_0 > 1$ such that $|\Im w| < \varepsilon$ for all $w = u+iv \in \partial D$ such that $u \ge u_0$. For $u_1 > u_0$, let H^- be the right half-plane with respect to $l_{\pi/4}(u_1)$. If u_1 is large enough, the set $D \cap H^-$ lies in the horizontal strip between $h(-\varepsilon)$ and $h(\varepsilon)$ and therefore the polarization D_p of D w.r.t. $l_{\pi/4}(u_1)$ lies in the vertical strip between $l(0)$ and $l(u_1 + \varepsilon)$. Let D^* be the connected component containing $w = 1$ of the Steiner symmetrization of D_p w.r.t. $\mathbb{R}$. Note that D^* is simply connected. Let $F(z) = 1 + \beta z + a_2(F)z^2 + \dots$, $\beta > 0$ map $\mathbb{U}$ conformally onto D^*. Note that $\text{Area}\, D^* \le \text{Area}\, D$ and $\beta > \alpha$ by the principles of symmetrization and polarization [16, 11, 24]. Therefore $F_\tau(z) = F(\tau z)$ with $\tau = \alpha/\beta$ is in P_α and satisfies (2.3) contradicting the extremality of f.

It is clear from geometry that an extremal domain has no slits, this can be also shown by using (2.3). Since D is bounded, starlike w.r.t. $w = 1$, circularly symmetric w.r.t. $\mathbb{R}_1$ and ∂D does not contain slits, it follows [4, Lemma 4] that D is a Jordan domain. Since $\arg(w - 1)$ and $|w - 1|$ are monotone when w runs along the boundary arc $\partial D \cap \mathbb{H}$, it follows that ∂D is rectifiable.

Let us show that ∂D contains 0. If not, then $\bar{D} \subset \mathbb{H}_r$. This implies that $f_\tau(z) = 1 + \tau^{-1}(f(\tau z) - 1)$ is in P_α for $\tau < 1$ close enough to 1. Since f is not linear,

$$D(f_\tau) = \alpha^2 + \sum_{n=2}^{\infty} n\tau^{2n-2} |a_n(f)|^2 < \alpha^2 + \sum_{n=2}^{\infty} n|a_n(f)|^2 = D(f),$$

contradicting the extremality of f. $\square$

Using the fact that $0 \in \partial D$ for the extremal domains, we can show that $A(\alpha)$ is strictly convex for $1 < \alpha < 2$. Indeed, let f_1, f_2 be extremal for P_{α_1} and P_{α_2} with $1 \le \alpha_1 < \alpha_2 < 2$.

If $A(\alpha) = (1/2)(A(\alpha_1) + A(\alpha_2))$ with $\alpha = (\alpha_1 + \alpha_2)/2$, then (2.1) holds with the sign of equality and therefore $f_2' = (\alpha_2/\alpha_1)f_1'$, which implies that $f_2 = (\alpha_2/\alpha_1)(f_1 - 1) + 1$. The latter is impossible since $\Re\left((\alpha_2/\alpha_1)(f_1(z) - 1) + 1\right)$ changes its sign in $\mathbb{U}$ if $f_1(\mathbb{U})$ has the point $w = 0$ on the boundary.

Now we show that Lavrent'ev's problem for pairs of analytic functions with nonoverlapping images covering fixed area can be reduced to the minimal area problem for P_α. For this we shall use a conformal variant of the averaging transformation of M. Marcus [20, 11]. By the Marcus transformation of domains D_1 and D_2 containing z_0 we mean a domain D^* starlike with respect to z_0 and such that

$$\lambda\left((D^* \setminus \mathbb{U}_\varepsilon(z_0))) \cap \gamma_\theta(z_0)\right) =$$

$$\left(\lambda\left(D_1 \setminus \mathbb{U}_\varepsilon(z_0)\right) \cap \gamma_\theta(z_0)) \cdot \lambda\left(D_2 \setminus \mathbb{U}_\varepsilon(z_0)\right) \cap \gamma_\theta(z_0))\right)^{1/2}$$

for all $\theta \in \mathbb{R}$ and $\varepsilon > 0$ such that $\mathbb{U}_\varepsilon(z_0) \subset D_1 \cap D_2$.

Let $D_1 \ni 1$ and $D_2 \ni -1$ be nonoverlapping domains on $\overline{\mathbb{C}}$ and let Ω_1 and Ω_2 be images of D_1 and D_2 under the mappings $\varphi : z \mapsto (z-1)/(z+1)$ and $1/\bar{\varphi}$, respectively. Let Ω^* be the Marcus transformation of Ω_1 and Ω_2 with respect to the origin.

By the averaging of D_1 and D_2 with base points 1 and -1 we mean a pair of domains

$$D_1^* = \varphi^{-1}(\Omega^*) \quad \text{and} \quad D_2^* = \{z :\, -\bar{z} \in D_1^*\}.$$

This transformation has already appeared a few times in the literature, see [23]. For the Marcus transformation it is known that [20, 11]

$$R(D_1, z_0)R(D_2, z_0) \le R^2(D^*, z_0) \quad \text{and} \quad \operatorname{Area} D_1 + \operatorname{Area} D_2 \ge 2\operatorname{Area} D^*. \tag{2.4}$$

Equality in the first inequality in (2.4) occurs if and only if $D_2 = \{z = z_0 + t(z - z_0) : z \in D_1\}$ with $t > 0$ up to a set of zero logarithmic capacity and in the second inequality if and only if $D_1 = D_2$ up to a set of zero area.

Similar inequalities with the same conditions of equality hold true for the averaging transformation of D_1, D_2 with two base points.

Lemma 3 *Let functions f_1 and f_2 be extremal for Lavrent'ev's problem for analytic functions with nonoverlapping images covering fixed area $2S$ and normalized by $f_1(0) = 1$, $f_2(0) = -1$. Then $D_1 = f_1(\mathbb{U})$ and $D_2 = f_2(\mathbb{U})$ are symmetric to each other with respect to the imaginary axis and D_1 is extremal for the minimal area problem in P_α with $\alpha = A^{-1}(S)$, where A^{-1} denotes the inverse of the area functional A.*

The functions f_1, f_2 map $\mathbb{U}$ conformally onto D_1 and D_2 respectively.

Proof. Existence of extremals is a consequence of the semicontinuity of the area. Assume that D_1 and D_2 are not symmetric to each other w.r.t. $l(0)$ up to a set of zero area. Let D_1^*, D_2^* be the averaging transformation of D_1, D_2 with base points 1 and -1.

Note that D_1^* and D_2^* are simply connected. Let F_1 and F_2 be conformal mappings of $\mathbb{U}$ onto D_1^* and D_2^* such that $F_1(0) = 1$, $F_2(0) = -1$, $F_1'(0) = F_2'(0) = \beta > 0$.

By the principle of symmetrization for the considered transformations [11],

$$|f_1'(0)f_2'(0)| \le R(D_1, 1)R(D_2, -1) \le R^2(D_1^*, 1) = \beta^2. \tag{2.5}$$

Besides, $\operatorname{Area} D_1^* < S$, since D_1 and D_2 are not symmetric to each other with respect to $l(0)$ up to a set of zero area. Since $1 \in D_1^* \subset \mathbb{H}_r$ and $\operatorname{Area} D_1^* < S$, Theorem 1 implies that

$$\beta < A^{-1}(S). \tag{2.6}$$

Inequalities (2.5), (2.6) contradict the assumptions on the extremality of f_1 and f_2 for the problem under consideration.

If D_1 and D_2 are symmetric to each other with respect to $l(0)$ up to a set of zero area, then $1 \in D_1 \subset \mathbb{H}_r$. Now the desired assertion follows from Theorem 1. $\Box$

The Marcus averaging transformation provides another way to prove uniqueness of the extremal function in P_α. Assume that there are two different extremal domains $D_1 = f_1(\mathbb{U})$ and $D_2 = f_2(\mathbb{U})$. Let D be the Marcus averaging transformation of D_1 and D_2 w.r.t. $w = 1$. By (2.3),

$$R(D, 1) = \beta > \alpha \quad \text{and} \quad \operatorname{Area} D < (1/2)(\operatorname{Area} D_1 + \operatorname{Area} D_2) = A(\alpha).$$

If F maps $\mathbb{U}$ conformally onto D such that $F(0) = 1$, $F'(0) = \beta$, then $F_{\alpha/\beta}$ is in P_α and by (2.1), $D(F_{\alpha/\beta}) < D(f_1)$ contradicting extremality of f_1.

To study the boundary behavior of f', we shall apply the polarization technique developed in [24]. Let H_τ^+ and H_τ^- be the left and right half planes with respect to the horizontal line $h(\tau)$. For $D \subset \overline{\mathbb{C}}$, let $D_\tau^+ = D \cap H_\tau^+$, $D_\tau^- = D \cap H_\tau^-$ and let D_τ^* denote the set symmetric to D w.r.t. $h(\tau)$.

We say that D possesses the polarization property in the interval $\tau_1 < \tau < \tau_2$ if $(D_\tau^+)^* \subset D_\tau^-$ for all $\tau_1 < \tau < \tau_2$. The following two lemmas are limit cases of Theorem 1 in [24].

Lemma 4 *Let f map $\mathbb{U}$ conformally onto a simply connected domain D and let f map a boundary arc $\{e^{i\theta} : \theta_1 < \theta < \theta_2\}$ onto a vertical interval $\{w : \Re w = u_0, \tau_1 < \Im w < \tau_2$. Let $\Im f(0) \le \tau_1$ and let D possess the polarization property in $\tau_1 < \tau < \tau_2$. Then $|f'(e^{i\theta})|$ strictly increases in $\theta_1 < \theta < \theta_2$ if $f(e^{i\theta_1}) = u_0 + i\tau_1$ and strictly decreases if $f(e^{i\theta_1}) = u_0 + i\tau_2$.*

Proof. Assuming that $f(e^{i\theta_1}) = u_0 + i\tau_1$, let $\theta_1 < \theta' < \theta'' < \theta_2$ and let $f(e^{i\theta'}) = w' = u_0 + i\tau'$, $f(e^{i\theta''}) = w'' = u_0 + i\tau''$. Since f preserves orientation we have $\tau_1 < \tau' < \tau'' < \tau_2$.

Let $g(w)$ denote Green's function of D with a pole at $w = f(0)$. Then

$$g(w) = -\log|f^{-1}(w)|. \tag{2.7}$$

Let $\varepsilon > 0$ be such that the discs $\mathbb{U}_\varepsilon(w' - \varepsilon)$ and $\mathbb{U}_\varepsilon(w'' - \varepsilon)$ are in D. Then the function

$$U(w) = g(w) - g(w^*),$$

where w^* is symmetric to w w.r.t. the line $h(\tau_0) = h((\tau' + \tau'')/2)$, is harmonic in the disc $\mathbb{U}_\varepsilon(w' - \varepsilon)$ and continuous in its closure.

Since D possesses polarization symmetry w.r.t. $h(\tau_0)$, $U(w) > 0$ in $\mathbb{U}_\varepsilon(w' - \varepsilon)$, by Theorem 1 in [24]. Since $U(w') = g(w') - g(w'') = 0$, U takes the minimal value at $w = w'$. Therefore by Hopf's lemma, see [22],

$$\frac{\partial g(w')}{\partial u} - \frac{\partial g(w'')}{\partial u} = \frac{\partial}{\partial u} U(w') < 0, \quad \text{where} \quad w = u + iv. \tag{2.8}$$

Since $zf'(z) > 0$ for $z = e^{i\theta}$ with $\theta_1 < \theta < \theta_2$, using (2.7) we get

$$\frac{\partial}{\partial u} g(u_0 + i\tau) = -\frac{\partial}{\partial u} \log |f^{-1}(w)| \,\big|_{w=u_0+i\tau} = -\Re \left((zf'(z))^{-1} \right)\Big|_{z=e^{i\theta}} = -|f'(e^{i\theta})|^{-1} \tag{2.9}$$

for all $\theta_1 < \theta < \theta_2$. (2.8) and (2.9) imply that $|f'(e^{i\theta'})| < |f'(e^{i\theta''})|$. This proves monotonicity in the case under consideration. The case $f(e^{i\theta_1}) = u_0 + i\tau_2$ is treated similarly. $\Box$

Let $\gamma_\varphi = \gamma_\varphi(0)$. We say that a domain D possesses the angular polarization property in $\varphi_1 < \varphi < \varphi_2$ if $(D^+_\varphi)^* \subset D^-_\varphi$ for all $\varphi_1 < \varphi < \varphi_2$. Here $D^+_\varphi = H^+_\varphi \cap D$, $D^-_\varphi = D \setminus \overline{D^+_\varphi}$, and H^+_φ denotes the left half-plane with respect to γ_φ, and $(\cdot)^*$ denotes a set symmetric to a given set with respect to γ_φ.

The following lemma is not needed for what follows and is included here for future use.

Lemma 5 *Let f map $\mathbb{U}$ conformally onto D and map a boundary arc $\{e^{i\theta} : \theta_1 < \theta < \theta_2\}$ onto a circular arc $L = \{w = \rho e^{i\varphi} : \varphi_1 < \varphi < \varphi_2\}$. Let $f(0) \in H^-_{\varphi_1} \cap H^-_{\varphi_2}$ and let D possesses the angular polarization property in $\varphi_1 < \varphi < \varphi_2$.*

Then $|f'(e^{i\theta})|$ strictly increases in $\theta_1 < \theta < \theta_2$ if $f(e^{i\theta_1}) = \rho e^{i\varphi_1}$ and strictly decreases in $\theta_1 < \theta < \theta_2$ if $f(e^{i\theta_1}) = \rho e^{i\varphi_2}$.

Proof repeats the proof of Lemma 4.

The extremal domain $D = f(\mathbb{U})$ with $f \in P_\alpha$ has Steiner symmetry and therefore possesses the polarization property for all $\tau > 0$. Thus we obtain from Lemma 4.

Corollary 2 *Let f be extremal in P_α, $1 < \alpha < 2$. If $\Re f(e^{i\theta}) = 0$ for $\theta_1 < \theta < 2\pi - \theta_1$ with $0 < \theta_1 < \pi$, then $|f'(e^{i\theta})|$ strictly decreases in $\theta_1 < \theta < \pi$ and strictly increases in $\pi < \theta < 2\pi - \theta_1$.*

3 Local variations and boundary derivatives

We derive variational formulas for the conformal radius that can be used to prove regularity a.e. of the free boundary of the minimal area problems. First we consider two well known elementary variations of the unit disk.

Lemma 6 *For $0 < \varepsilon < 1/2$, let $U^\varepsilon = \mathbb{U} \setminus \overline{\mathbb{U}}_\varepsilon(1-\varepsilon)$. Then*

$$R(U^\varepsilon, 0) = \frac{1-\varepsilon}{\pi\varepsilon} \sin\frac{\pi\varepsilon}{1-\varepsilon} = 1 - \frac{\pi^2}{6}\varepsilon^2 + O(\varepsilon^3) \quad \textit{as} \quad \varepsilon \to 0. \tag{3.1}$$

Proof. The function

$$f(z) = \exp\left\{\frac{\pi i\varepsilon}{1-\varepsilon}\frac{1+z}{1-z}\right\}$$

maps U^ε conformally onto the upper half plane $\mathbb{H}$. Since

$$R(U^\varepsilon, 0)|f'(0)| = R(H, f(0)) = 2\Im f(0),$$

and

$$f(0) = \exp\left\{\frac{\pi i\varepsilon}{1-\varepsilon}\right\}, \qquad f'(0) = \frac{2\pi i\varepsilon}{1-\varepsilon}\exp\left\{\frac{\pi i\varepsilon}{1-\varepsilon}\right\},$$

we get (3.1). $\square$

For $0 < \alpha \leq \pi$, let $L(\alpha) = \{e^{i\theta} : |\theta| < \pi\}$ and $L^-(\alpha) = \mathbb{T} \setminus \overline{L(\alpha)}$.

Lemma 7 *Let $0 < \varepsilon < 1/2$ and $-\frac{1}{2} \leq \varphi \leq \frac{1}{2}$. Let $U^{\varepsilon,\varphi}$ be a simply connected domain in $\mathbb{U}$ containing the origin and bounded by the arc $L^-(\varepsilon)$ and the circular arc $L(\varepsilon,\varphi)$ with ends at the points $e^{i\varepsilon}$ and $e^{-i\varepsilon}$ that forms an angle of opening $\pi\varphi$ with the arc $L(\varepsilon)$ at the point $e^{i\varepsilon}$. Then*

$$R(U^{\varepsilon,\varphi}, 0) = \frac{1-\varphi}{\sin\varepsilon} \sin\frac{\varepsilon}{1-\varphi} = 1 - \frac{\varphi(2-\varphi)}{6(1-\varphi)^2}\varepsilon^2 + O(\varepsilon^3) \quad \textit{as} \quad \varepsilon \to 0. \tag{3.2}$$

Proof. The function $f = f_2 \circ f_1$ with

$$f_1(z) = i\frac{1-z}{1+z} \quad \text{and} \quad f_2(\zeta) = \left(\frac{\zeta - \tan\varepsilon/2}{\zeta + \tan\varepsilon/2}\right)^{1/(1-\varphi)}$$

maps $U^{\varepsilon,\varphi}$ conformally onto the upper half-plane $\mathbb{H}$. By direct computation,

$$f(0) = \exp(\varepsilon i/(1-\varphi)) \quad \text{and} \quad f'(0) = 2\sin\varepsilon/(1-\varphi)$$

and therefore (3.2) follows from the equality $R(U^{\varepsilon,\varphi}, 0)|f'(0)| = 2\Im f(0)$. $\square$

Now we aim to extend the above variational technique to the case of simply connected domains that are conformal at the center of the considered boundary variation.

Let $w = f(z)$ map $\mathbb{H}$ conformally onto $D = f(\mathbb{H})$ such that $f(0) = 0$ and

$$\lim_{\mathbb{H}\ni z\to 0} \frac{f(z)}{z} = 1 \quad \text{or equivalently} \quad f(z) = z + \alpha(z)z \tag{3.3}$$

with $\alpha(z) \to 0$ as $z \to 0$ in $\overline{\mathbb{H}}$. We assume below that in a neighborhood of the origin the boundary ∂D is Jordan and rectifiable.

For $\varepsilon > 0$ small enough, let c^o_ε and c_ε be open and closed crosscuts of D at the boundary point $w = 0$, i.e. c^o_ε and c_ε are respectively the biggest open and closed arcs of $C_\varepsilon := C_\varepsilon(0)$ such that $i\varepsilon \in c^o_\varepsilon \subset D$ and $i\varepsilon \in c_\varepsilon \subset \overline{D}$, respectively. For $r > 0$, let $\mathbb{U}_r = \mathbb{U}_r(0)$, $\mathbb{U}^+_r = \mathbb{U}_r \cap \mathbb{H}$. Let $\hat{D}_\varepsilon$ be a connected component of $D \setminus \overline{\mathbb{U}}_\varepsilon$ containing $f(i)$ and let

$$D_\varepsilon = \hat{D}_\varepsilon \cup \mathbb{U}^+_\varepsilon \cup c^o_\varepsilon. \tag{3.4}$$

Then D_ε is a simply connected domain having the segment $[-\varepsilon, \varepsilon]$ on its boundary. Let $w = f_\varepsilon(z)$ map $\mathbb{H}$ conformally onto D_ε such that

$$f_\varepsilon(i) = f(i) = a, \qquad f_\varepsilon(0) = 0.$$

Let $c^o_{\varepsilon,\delta}$ and $c_{\varepsilon,\delta}$ be the open and closed crosscuts of D_ε corresponding to the circle C_δ. The varied domain D_ε is shown in Figure 3, which also demonstrates some notations used in the proof of Lemma 8.

Lemma 8 *If* $z \in \Gamma_\varepsilon = \{z \in \overline{\mathbb{H}} : \ f_\varepsilon(z) \in c_{\varepsilon,\varepsilon}\}$, *then*

$$\varepsilon(1 - \alpha(\varepsilon)) \le |z| \le \varepsilon(1 + \alpha(\varepsilon)) \tag{3.5}$$

with $\alpha(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof. For any decreasing sequence $\varepsilon_n \to 0$ of positive ε_n, consider a sequence of simply connected domains $\Omega_n = \{w : \ \varepsilon_n w \in D_{\varepsilon_n}\}$. (3.3) implies that Ω_n converges to the kernel $\mathbb{H}$. Let

$$g_n(z) = \varepsilon_n^{-1} f_{\varepsilon_n}(|f^{-1}_{\varepsilon_n}(i\varepsilon_n)|z)$$

and let $\tau_n = g_n^{-1}(i)$. Then $|\tau_n| = 1$. Let us show that $\tau_n \to i$.

If $\zeta_0 \neq 0$ is a single boundary point of D, then for all ε_n small enough, $c^o_{\varepsilon_n}$ separates ζ_0 from 0 in D_{ε_n}. We split $\partial\Omega_n$ into six arcs γ^k_n, $k = \pm1, \pm2, \pm3$, where $\gamma^1_n =]0, 1[$, $\gamma^{-1}_n =]-1, 0[$, $\gamma^2_n \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{U}}$ and joins the point $\varepsilon_n^{-1}\zeta_0$ with some point on $\mathbb{T}$ that is close to $w = 1$. Similarly, $\gamma^{-2}_n \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{U}}$ and joins $\varepsilon_n^{-1}\zeta_0$ with some point of $\mathbb{T}$ that is close to $w = -1$. The arcs γ^3_n and γ^{-3}_n which may consist of a single point, are complementary to the others such that γ^3_n and γ^{-3}_n lie in vicinities of the points $w = 1$ and $w = -1$, respectively.

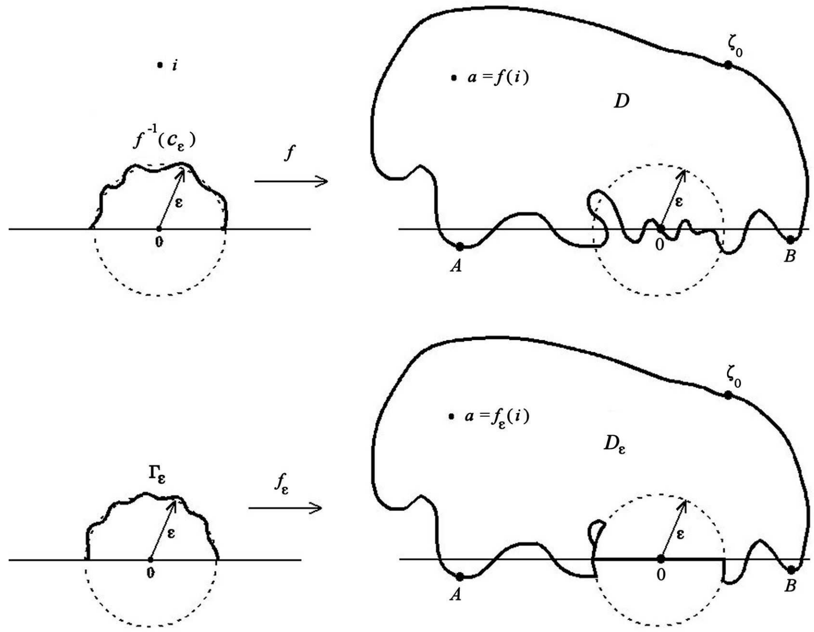

Figure 3: Variation D_ε

Let $\omega_n^k = \omega(i, \gamma_n^k, \Omega_n)$ denote the harmonic measure of γ_n^k w.r.t. Ω_n at $w = i$. Standard estimates for the harmonic measure show that

$$\omega_n^k \to 1/4 \quad \text{for} \quad k = \pm 1, \pm 2 \quad \text{and} \quad \omega_n^{\pm 3} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.6}$$

Let $l_n^k = g_n^{-1}(\gamma_n^k)$ and $x_n = g_n^{-1}(\varepsilon_n^{-1}\zeta_0)$. Then

$$x_n \to \infty \quad \text{as} \quad n \to \infty. \tag{3.7}$$

Since harmonic measure is conformally invariant, $\omega(\tau_n, l_n^k, \mathbb{H}) = \omega_n^k$. This combined with (3.6), (3.7) and the equality $|\tau_n| = 1$ implies that $\tau_n \to i$. And what is more, the above arguments show that l_n^1, l_n^{-1}, l_n^2, and l_n^{-2} converge to intervals $]0, 1[$, $]-1, 0[$, $]1, +\infty[$, and $]-\infty, -1[$, respectively. Therefore, l_n^3 and l_n^{-3} shrink respectively to the points $z = 1$ and $z = -1$.

Since $\tau_n \to i$, $g_n(0) = 0$, and Ω_n converges to the kernel $\mathbb{H}$, the Carathéodory convergence theorem implies that

$$g_n(z) \to z \tag{3.8}$$

uniformly on compact subsets of $\mathbb{H}$.

From (3.8) and (3.6) we derive by contradiction that for $z \in \Gamma_{\varepsilon_n}$,

$$\rho(n)(1 - \alpha_n) \le |z| \le \rho(n)(1 + \alpha_n) \tag{3.9}$$

with some $\rho(n) > 0$ and $\alpha_n \to 0$ as $n \to \infty$.

To specify $\rho(n)$, we apply a technique based on the properties of the reduced modulus of a triangle, see [25]. We fix distinct points A and B on $\partial D \setminus \{0\}$ and consider D and D_{ε_n} as triangles having vertices at 0, A and B. Let L denote a side of D (or D_{ε_n}) between A and B. For $0 < \delta < \varepsilon_n$, let $D(\delta)$ and $D_{\varepsilon_n}(\delta)$ denote components of $D \setminus c_\delta^o$ and $D_{\varepsilon_n} \setminus c_\delta^o$ containing $a = f(i)$. These components will be considered as quadrilaterals having a pair of arcs L and c_δ^o as their distinguished sides.

Let mod $(\cdot)$ denote the modulus of a quadrilateral w.r.t. the family of curves separating its distinguished sides, see [17, 25]. Then the reduced modulus of the triangles D and D_{ε_n} w.r.t. the vertex $w = 0$ is defined by the following limits:

$$m(D; 0|A, B) = \lim_{\delta \to 0} (\operatorname{mod}(D(\delta)) + (1/\pi) \log \delta), \tag{3.10}$$

$$m(D_{\varepsilon_n}; 0|A, B) = \lim_{\delta \to 0} (\operatorname{mod}(D_{\varepsilon_n}(\delta)) + (1/\pi) \log \delta), \tag{3.11}$$

which exist and are finite since f and f_{ε_n} have angular derivatives at $z = 0$, see [25, Theorem 1.3].

Let $a = f^{-1}(A)$, $b = f^{-1}(B)$, $a_n = f_{\varepsilon_n}^{-1}(a)$, $b_n = f_{\varepsilon_n}^{-1}(b)$. Considering $\mathbb{H}$ as a triangle with corresponding vertices and using the usual formula for the change in the reduced modulus under conformal mapping [25, Lemma 1.3] we get

$$m(D; 0|A, B) = m(\mathbb{H}; 0|a, b) + (1/\pi) \log |f'(0)|, \tag{3.12}$$

$$m(D_{\varepsilon_n}; 0|A, B) = m(\mathbb{H}; 0|a_n, b_b) + (1/\pi) \log |f'_{\varepsilon_n}(0)|. \tag{3.13}$$

Note that $a_n \to a$, $b_n \to b$ since f_{ε_n} converges to f as $n \to \infty$. This implies

$$m(\mathbb{H}; 0|a_n, b_n) \to m(\mathbb{H}; 0|a, b) \quad \text{as} \quad n \to \infty. \tag{3.14}$$

For $0 < \delta < \varepsilon_n$, let $G_1(n, \delta)$ and $G_2(n, \delta)$ be the quadrilaterals cut off from $D(\delta)$ by the closed crosscut c_{ε_n}. Let γ_A and γ_B be arcs of $\partial D \setminus (L \cup \overline{\mathbb{U}}_{\varepsilon_n})$ joining the vertices A and B with c_{ε_n} and let γ_+ and γ_- be arcs of $\partial D \cap \mathbb{U}_{\varepsilon_n}$ joining the ends of c_δ^o with c_{ε_n}. We assume that the quadrilaterals $G_1(n, \delta)$ and $G_2(n, \delta)$ have the arcs γ_A, γ_B and γ_+, γ_- as a pair of their (non-distinguished) sides respectively. These quadrilaterals and corresponding notations are shown in the logarithmic coordinates in Figure 4.

Let $\widehat{G}_1(n.\delta)$, $\widehat{G}_2(n, \delta)$ be similar quadrilaterals cut off by c_{ε_n} from $D_{\varepsilon_n}(\delta)$, see Figure 4. Note that $G_1(n, \delta) = \widehat{G}_1(n, \delta)$ and $\widehat{G}_2(n, \delta)) = \{w : \delta < |w| < \varepsilon_n, 0 < \arg w < \pi\}$. Therefore,

$$\operatorname{mod}(G_1(n, \delta)) = \operatorname{mod}(\widehat{G}_1(n, \delta)), \quad \operatorname{mod}(\widehat{G}_2(1, \delta)) = (1/\pi) \log(\varepsilon_n/\delta). \tag{3.15}$$

It follows from the boundary conformality criterion proved by Rodin and Warshawski and independently by Jenkins and Oikawa, see [21, Theorem 11.9] that

$$\operatorname{mod}(G_2(n, \delta)) = (1/\pi) \log(\varepsilon_n/\delta) + \alpha_0(\varepsilon_n, \delta) \tag{3.16}$$

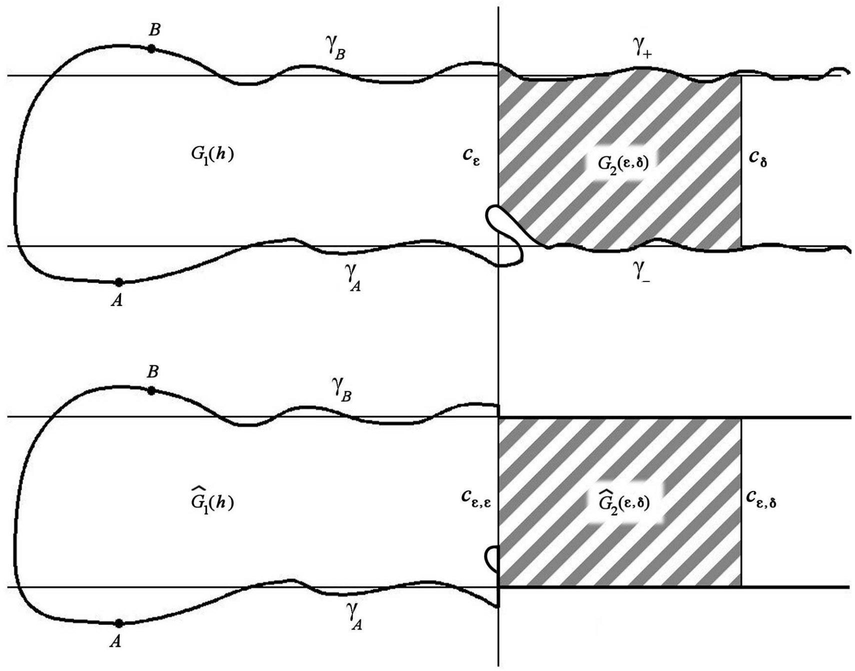

Figure 4: Decomposition into quadrilaterals

with $|\alpha_0(\varepsilon_n, \delta)| \le \omega_0(\varepsilon_n)$ for all $0 < \delta \le \varepsilon_n$, where $\omega_0(\varepsilon_n) \to 0$ as $\varepsilon_n \to 0$.

Relations (3.3) and (3.9) imply that $\operatorname{mod}(D(\delta))$ and $\operatorname{mod}(D_{\varepsilon_n}(\delta))$ are "approximately additive', cf. [21, Proposition 11.8], i.e.,

$$\operatorname{mod}(D(\delta)) = \operatorname{mod}(G_1(n,\delta)) + \operatorname{mod}(G_2(n,\delta)) + \alpha_1(\varepsilon_n, \delta), \tag{3.17}$$

$$\operatorname{mod}(D_{\varepsilon_n}(\delta)) = \operatorname{mod}(\widehat{G}_1(n,\delta)) + \operatorname{mod}(\widehat{G}_2(n,\delta)) + \alpha_2(\varepsilon_n, \delta), \tag{3.18}$$

where $|\alpha_1(\varepsilon_n, \delta)| + |\alpha_2(\varepsilon_n, \delta)| \le \omega_1(\varepsilon_n)$ for all $0 < \delta \le \varepsilon_n$ with $\omega_1(\varepsilon_n) \to 0$ as $\varepsilon_n \to 0$.

Combining relations (3.10)–(3.18), we conclude that

$$|f'_{\varepsilon_n}(0)| \to |f'(0)| = 1 \quad \text{as} \quad \varepsilon_n \to 0. \tag{3.19}$$

Combining (3.19) and (3.9) with the second equality in (3.15), we deduce that (3.9) holds with $\rho(n) = \varepsilon_n$. Since ε_n is an arbitrary vanishing sequence, (3.5) follows. □

Corollary 3 *$m(D_\varepsilon, a) = m(D, a) + o(\varepsilon^2)$ as $\varepsilon \to 0$.*

Proof. Let $\Omega_\varepsilon = f^{-1}(\hat{D}_\varepsilon)$, $\Omega^\varepsilon = f_\varepsilon^{-1}(\hat{D}_\varepsilon)$ and let $G_\varepsilon = \varphi(\Omega_\varepsilon)$, $G^\varepsilon = \varphi(\Omega^\varepsilon)$, where $\varphi(w) = (i - w)/(i + w)$ maps $\mathbb{H}$ onto $\mathbb{U}$. The formula for the change in the reduced modulus under conformal mapping implies that

$$m(D_\varepsilon, a) - m(D, a) = m(\Omega^\varepsilon, i) - m(\Omega_\varepsilon, i) = m(G^\varepsilon, 0) - m(G_\varepsilon, 0). \tag{3.20}$$

Since $\varphi'(0) = 2i$, (3.3) and (3.5) show that

$$U^{\delta',\pi/2} \subset G_\varepsilon,\ G^\varepsilon \subset U^{\delta'',\pi/2}, \tag{3.21}$$

where $\delta' = 2\varepsilon(1-\alpha(\varepsilon))$, $\delta'' = 2\varepsilon(1+\alpha(\varepsilon))$ with $0 \le \alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Since the reduced modulus is a monotone function of the domain, (3.21) and (3.2) with $\varphi = \pi/2$ yield

$$m(G^\varepsilon, 0) - m(G_\varepsilon, 0) = o(\varepsilon^2),$$

which combined with (3.20) implies the desired assertion. □

Let $\Delta_\varepsilon^+ = f_\varepsilon^{-1}(\mathbb{U}_\varepsilon^+)$, $\Delta_\varepsilon^- = \{z : \bar{z} \in \Delta_\varepsilon^+\}$. By the symmetry principle f_ε can be continued up to the conformal mapping from $\hat{H}_\varepsilon = \mathbb{H} \cup \Delta_\varepsilon^- \cup]f_\varepsilon^{-1}(-\varepsilon), f_\varepsilon^{-1}(\varepsilon)[$ onto the domain $\hat{D}_\varepsilon = D_\varepsilon \cup \mathbb{U}_\varepsilon$. Let $\gamma(\varepsilon,\varphi)$ denote the circular arc through the points $\pm\varepsilon$ that forms the angle $\pi\varphi$, $-1/2 \le \varphi \le 1/2$, with the segment $[-\varepsilon,\varepsilon]$ at $-\varepsilon$. Let $M(\varepsilon,\varphi)$ denote the lune bounded by $[-\varepsilon,\varepsilon]$ and $\gamma(\varepsilon,\varphi)$ and let $M(\varepsilon_1,\varepsilon_2,\varphi) = M(\varepsilon_2,\varphi) \setminus \overline{M(\varepsilon_1,\varphi)}$ with $0 < \varepsilon_1 < \varepsilon_2$.

Corollary 4 *Let $l(\varepsilon,\varphi) = f_\varepsilon^{-1}(\gamma(\varepsilon,\varphi))$. Then*

$$l(\varepsilon,\varphi) \subset M(\varepsilon(1-\alpha), \varepsilon(1+\alpha), \varphi) \tag{3.22}$$

with $\alpha = \alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. Assume that (3.22) is not valid for some sequence $\varepsilon_n \to 0$. Then without loss of generality we may assume that there is a number k, $0 < k < 1$ and a sequence of points $w_n \in \gamma(\varepsilon_n,\varphi)$ such that $w_n/\varepsilon_n \to w_0$ and $z_n/\varepsilon_n \to z_0$ with $z_0 \notin M(k,k^{-1},\varphi)$, where $z_n = f_{\varepsilon_n}^{-1}(w_n)$.

Lemma 8 yields that the sequences $\psi_n(w) = \varepsilon_n^{-1} f_{\varepsilon_n}^{-1}(\varepsilon_n w)$ and $\psi_n^{-1}(z)$ converge to the identity mapping uniformly on compact subsets of $\mathbb{H}$. This implies that $z_0 = w_0$ if $z_0 \ne \pm 1$ or $w_0 \ne \pm 1$. Therefore $z_0 \in \gamma(1,\varphi) \subset M(k,k^{-1},\varphi)$ in this case. If $z_0 = 1$ or $z_0 = -1$ then $z_n = \varepsilon_n(1+\alpha_n)$ or $z_n = -\varepsilon_n(1+\alpha_n)$ with $\alpha_n \to 0$ as $n \to \infty$ and therefore (3.22) is satisfied. □

Corollary 5 *For a fixed $0 \le \varphi \le \pi/2$ and $\varepsilon > 0$ small enough, let*

$$D_{\varepsilon,\varphi} = D_\varepsilon \cup M(\varepsilon,\varphi) \cup]-\varepsilon,\varepsilon[.$$

Then

$$m(D_{\varepsilon,\varphi}, a) = m(D,a) + \frac{2\varphi(2+\varphi)}{3\pi(1+\varphi)^2}\varepsilon^2 + o(\varepsilon^2) \quad \textit{as} \quad \varepsilon \to 0. \tag{3.23}$$

Proof. By Corollary 3,

$$m(D_{\varepsilon,\varphi},a)-m(D,a)=m(D_{\varepsilon,\varphi},a)-m(D_\varepsilon,a)+o(\varepsilon^2).$$

By the symmetry principle, f_ε^{-1} maps $D_{\varepsilon,\varphi}$ conformally onto a domain $\Omega_{\varepsilon,\varphi}=f_\varepsilon^{-1}(D_{\varepsilon,\varphi})$. Corollary 4 implies

$$H^{\varepsilon',\varphi}\subset\Omega_{\varepsilon,\varphi}\subset H^{\varepsilon'',\varphi} \tag{3.24}$$

with $\varepsilon'=\varepsilon(1-\alpha(\varepsilon))$, $\varepsilon''=\varepsilon(1+\alpha(\varepsilon))$, $0\le\alpha(\varepsilon)\to0$ as $\varepsilon\to0$. Here $H^{\varepsilon,\varphi}=\mathbb{H}\cup M(\varepsilon,\varphi)\cup]-\varepsilon,\varepsilon[$.

Let $\psi(w)=(i-w)/(i+w)$ and let $G_{\varepsilon,\varphi}=\psi(\Omega_{\varepsilon,\varphi})$. The formula for the change in the reduced modulus gives

$$m(D_{\varepsilon,\varphi},a)-m(D_\varepsilon,a)=m(G_{\varepsilon,\varphi},0)-m(\mathbb{U},0)=m(G_{\varepsilon,\varphi},0). \tag{3.25}$$

Since $\psi'(0)=2i$, (3.24) implies that

$$U^{2\varepsilon',\varphi}\subset G_{\varepsilon,\varphi}\subset U^{2\varepsilon'',\varphi}.$$

Since the reduced modulus is a monotone function of a domain, the latter combined with (3.2) and (3.25) yields (3.23). □

Next we define a local variation of a Jordan rectifiable domain Ω in vicinities of its two boundary points w_1 and w_2. Let Ω have a unit inward normal n_1 at a boundary point w_1. For $\varepsilon_1>0$ small enough let $\tilde\Omega_{\varepsilon_1}$ be the variation of the domain $\{w:\,w_1-in_1w\in\Omega\}$ defined by (3.4). For $0\le\varphi_1\le\pi/2$, let

$$\tilde\Omega(\varepsilon_1,\varphi_1)=\tilde\Omega_{\varepsilon_1}\cup M(\varepsilon_1,\varphi_1)\cup]-\varepsilon_1,\varepsilon_1[.$$

The domain

$$\Omega_{\varepsilon_1,\varphi_1}(w_1)=\{w:\,in_1(w-w_1)\in\tilde\Omega(\varepsilon_1,\varphi_1)\}$$

will be called a variation of Ω centered at w_1 with radius ε_1 and inclination φ_1.

Let $w_0\in\Omega$ and let $g(w,\varepsilon_1,\varphi_1)$ map $\Omega_{\varepsilon_1,\varphi_1}(w_1)$ conformally onto $\mathbb{U}$ such that $g(w_0)=0$, $g(w_2)=1$. Let $0<\varphi_2\le\pi/2$ and $\varepsilon_2>0$ be small enough. The domain

$$\tilde\Omega=g^{-1}(U^{\varepsilon_2,\varphi_2},\varepsilon_1,\varphi_1), \tag{3.26}$$

where $U^{\varepsilon_2,\varphi_2}$ is defined in Lemma 7, will be called *the two point variation of* Ω *centered at* w_1 *and* w_2 *with radii* ε_1 *and* ε_2 *and inclinations* φ_1 *and* φ_2. Varying domain $\tilde\Omega$ and corresponding notations are depicted in Figure 5. Later on φ_1 and φ_2 will be fixed and $\varepsilon_1\to0$, $\varepsilon_2\to0$.

To derive a variational formula for the area, we need a variant of the Carathéodory convergence theorem stated in Lemma 9 below.

Let Ω_k, $k=\infty,0,1,\dots$ be simply connected domains in $\mathbb{C}$ such that $\Omega_0\subset\Omega_k$ for all k and let each Ω_k contain an open Jordan arc $L\ni0$ on its boundary. Let the sequence Ω_k, $k=1,2,\dots$ converge to the kernel Ω_∞. Let f_k, $k=\infty,0,1,\dots$ map $\mathbb{H}$ conformally onto Ω_k such that $f_k(0)=0$, $f_k(i)=a\in\Omega_0$.

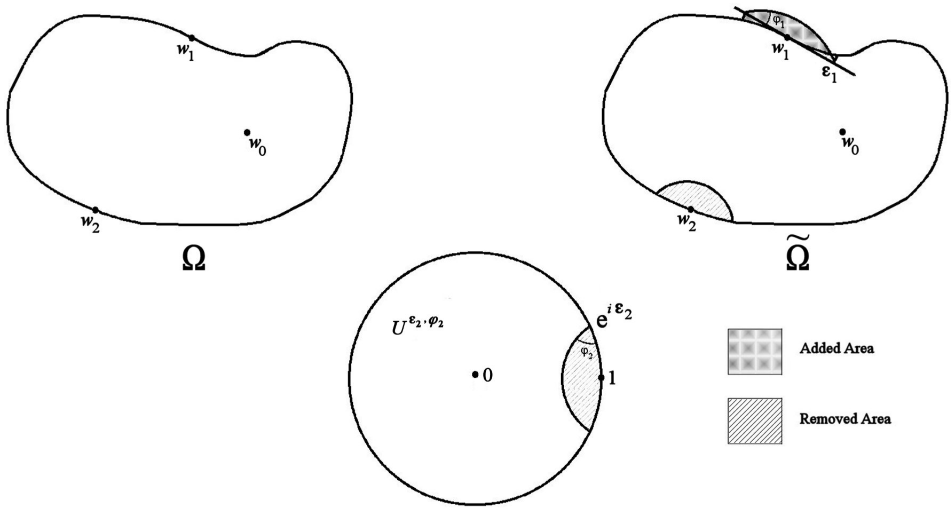

Figure 5: Two point variation $\widetilde{\Omega}$

Lemma 9 *If for $k = \infty$ there is a limit*

$$\lim_{\mathbb{H}\ni z\to 0} \frac{f_k(z)}{z} = f_k'(0) \neq 0, \infty, \tag{3.27}$$

then

a) the limit (3.27) exists for every $k = 0, 1, 2, \ldots$

b) $f_k'(0) \to f_\infty'(0)$ as $k \to \infty$;

c) the limit in a) is uniform in k, i.e., for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that if $z \in \overline{\mathbb{H}}$ and $|z| < \delta$, then

$$\left| \frac{f_k(z)}{z} - f_k'(0) \right| < \varepsilon \quad \textit{for all} \quad k. \tag{3.28}$$

Proof. a) Let $D_0 = f_\infty^{-1}(\Omega_0)$ and let ψ be a conformal mapping from $\mathbb{H}$ onto D_0 normalized by $\psi(i) = i$, $\psi(0) = 0$. Then $f_0 = f_\infty \circ \psi$. Since ψ is conformal and analytic at the origin, the latter shows that the existence of the limit (3.27) is a local characteristic of a domain at a vicinity of its boundary point at the origin and a) follows.

b) Let $\Omega = \overline{\mathbb{C}} \setminus L$ and let φ map Ω conformally onto $\mathbb{H}$ such that $\varphi(a) = i$ and $\varphi(0) = 0$. If $D_k = \varphi(\Omega_k)$, then D_k converges to the kernel D_∞ as $k \to \infty$. Let $g_k = \varphi \circ f_k$. Then $g_k(i) = i$ and $g_k'(0) = \varphi'(0) f_k'(0)$.

The function g_k can be continued analytically into the lower half-plane across a small interval $]-\delta, \delta[$ with $\delta > 0$ independent of k. Let $G(\delta) = \mathbb{C} \setminus ((-\infty, -\delta] \cup [\delta, +\infty))$. It is not difficult to see that $g_k(G(\delta))$ converges to the kernel $D_\infty \cup \{z : \bar{z} \in D_\infty\} \cup g_\infty(]-\delta, \delta[)$. Then the Carathéodory theorem implies, $g_k'(0) \to g_\infty'(0)$ as $k \to \infty$. Therefore, $f_k'(0) \to f_\infty'(0)$ as required.

c) Assume that there exist $\varepsilon_0 > 0$ and $z_k \in \overline{\mathbb{H}} \setminus \{0\}$, $z_k \to 0$ such that

$$\left| \frac{f_k(z_k)}{z_k} - f_k'(0) \right| \geq \varepsilon_0 \quad \text{for all} \quad k = 1, 2, \dots \tag{3.29}$$

Let $r_k = |z_k| > 0$. Consider functions $h_k(z) = (f_k'(0) r_k)^{-1} f_k(r_k z)$. We have

$$h_k(0) = 0, \quad h_k'(0) = \lim_{\mathbb{H} \ni z \to 0} (h_k(z)/z) = 1.$$

The sequence of domains $D_k = h_k(\mathbb{H})$ converges to the kernel $\mathbb{H}$. Let us show that $h_k(z) \to z$ uniformly on compact subsets of $\overline{\mathbb{H}}$. Consider functions

$$\tau_k(z) = (g_k'(0) r_k)^{-1} g_k(r_k z)$$

with g_k defined in b). It is easy to see that $\tau_k(z)$ continued to the lower half-plane as in b) converges to the identity mapping uniformly on compact subsets of $\mathbb{C}$. Since

$$h_k(z) = (f_k'(0) r_k)^{-1} \varphi^{-1}(g_k'(0) r_k \tau_k(z))$$

with φ defined in b), the latter implies that $h_k(z) \to z$ uniformly on compact subsets of $\overline{\mathbb{H}}$. Therefore, $h_k(e^{i\theta_k})/e^{i\theta_k} \to 1$ as $k \to \infty$ for every sequence θ_k, $0 \leq \theta_k \leq \pi$. This together with b) leads to

$$\left| \frac{f_k(z_k)}{z_k} - f_k'(0) \right| = |f_k'(0)| \left| \frac{h_k(e^{i\theta_k})}{e^{i\theta_k}} - 1 \right| \to 0$$

contradicting (3.29). $\square$

Lemma 10 *Let $w = f(z)$ map $\mathbb{U}$ conformally onto Ω defined above such that $f(0) = w_0$, $f(e^{i\theta_1}) = w_1$, $f(e^{i\theta_2}) = w_2$ and let there exist*

$$f'(e^{i\theta_k}) = \lim_{\overline{\mathbb{U}} \ni z \to e^{i\theta_k}} \frac{f(z) - w_k}{z - r^{i\theta_k}} \neq 0, \infty \quad \textit{for} \quad k = 1, 2. \tag{3.30}$$

Let $|f'(e^{i\theta_k})| = \alpha_k$, $k = 1, 2$. Let $\tilde{\Omega}(\varepsilon_1, \varepsilon_2, \varphi_1, \varphi_2)$ be the two point variation of Ω defined by (3.26) with ε_2 replaced by ε_2/α_2. Then for fixed $0 \leq \varphi_1 \leq \pi/2$ and $0 \leq \varphi_2 \leq \pi/2$,

$$m(\tilde{\Omega}(\varepsilon_1, \varepsilon_2, \varphi_1, \varphi_2), w_0) - m(\Omega, w_0) = \frac{2\varphi_1(2+\varphi_1)}{3\alpha_1^2(1+\varphi_1)^2}\varepsilon_1^2 - \frac{2\varphi_2(2-\varphi_2)}{3\alpha_2^2(1-\varphi_2)^2}\varepsilon_2^2 + o(\varepsilon_1^2) + o(\varepsilon_2^2) \tag{3.31}$$

and

$$\operatorname{Area}\tilde{\Omega}(\varepsilon_1, \varepsilon_2, \varphi_1, \varphi_2) - \operatorname{Area}\Omega = \frac{2\varphi_1 - \sin 2\varphi_1}{2\sin^2\varphi_1}\varepsilon_1^2 - \frac{2\varphi_2 - \sin 2\varphi_2}{2\sin^2\varphi_2}\varepsilon_2^2 + o(\varepsilon_1^2) + o(\varepsilon_2^2) \tag{3.32}$$

as $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

Proof. By the formula for the change in the reduced modulus and Lemma 7,

$$m(\tilde{\Omega}, w_0) - m(\Omega_{\varepsilon_1,\varphi_1}, w_0) = m(U^{\varepsilon_2/\alpha_2,\varphi_2}, 0) = -\frac{1}{2\pi}\frac{\varphi_2(2-\varphi_2)}{6(1-\varphi_2)^2}\frac{\varepsilon_2^2}{\alpha_2^2} + o(\varepsilon_2^2).$$

Using Corollary 5, we get

$$m(\Omega_{\varepsilon_1,\varphi_1}, w_0) - m(\Omega, w_0) = \frac{1}{2\pi}\frac{2\varphi_1(2+\varphi_1)}{3(1+\varphi_1)^2}\frac{\varepsilon_1^2}{\alpha_1^2} + o(\varepsilon_1^2).$$

Combining these relations we get (3.31).

Computation of the change in the area in a vicinity of the point w_1 requires only elementary geometry. To compute the change in the area in a vicinity of w_2, we use assertion c) of Lemma 9 combined with the same elementary computations. □

4 Boundary value problem for the extremals

In this section, f will denote the extremal function in P_α with $1 < \alpha < 2$ and $D = f(\mathbb{U})$. Let $L_f = \partial D \cap \mathbb{H}_r$ be the free boundary and $L_n = \partial D \setminus \bar{L}_f$. Then $\bar{L}_n = J(s)$, where $J(s) = [-is, is]$ with some $s \geq 0$. Let $l_f = f^{-1}(L_f)$, $l_n = \mathbb{T} \setminus \bar{l}_f$. We call l_f and l_n the free and nonfree arcs respectively.

Lemma 11 *f' is bounded in $\mathbb{U}$.*

Proof. Assume that $f'(z_k) \to \infty$ as $k \to \infty$ for a sequence $z_k \in \mathbb{U}$. Let φ_k denote a conformal mapping from $\mathbb{U}$ onto the domain $\mathbb{U} \setminus \overline{\mathbb{U}}_{\varepsilon_k}(z_k)$ with $\varepsilon_k = 1 - |z_k|$ normalized by $\varphi_k(0) = 0$, $\varphi_k'(0) > 0$ and let $f_k = f \circ \varphi_k$. Since $\varepsilon_k \to 0$ as $k \to \infty$, (3.1) implies that $f_k \in P_{\alpha_k}$ with

$$\alpha_k = \alpha(1 - \pi^2\varepsilon_k^2/6) + O(\varepsilon_k^3) \quad \text{as} \quad k \to \infty. \tag{4.1}$$

Using the mean value property we get

$$D(f_k) = \int_{\mathbb{U}} |f'|^2\, d\sigma - \int_{\mathbb{U}_{\varepsilon_k}(z_k)} |f'|^2\, d\sigma \leq A(\alpha) - \pi|f'(z_k)|^2\varepsilon_k^2. \tag{4.2}$$

Since $f_k \in P(\alpha_k)$ and $\alpha_k < \alpha$, (4.1) and (4.2) imply

$$A'(\alpha - 0) = \lim_{k\to\infty}\frac{A(\alpha_k) - A(\alpha)}{\alpha_k - \alpha} \geq \lim_{k\to\infty}\frac{D(f_k) - A(\alpha)}{\alpha_k - \alpha} \geq \pi\alpha^{-1}\lim_{k\to\infty}|f'(z_k)| = \infty,$$

contradicting the convexity property of $A(\alpha)$. □

Lemma 12 *There is $\beta > 0$ such that*

$$|f'(e^{i\theta})| = \beta \quad \textit{for a.e.} \quad e^{i\theta} \in l_f \quad \textit{and} \quad |f'(e^{i\theta})| < \beta \quad \textit{for all} \quad e^{i\theta} \in l_n. \tag{4.3}$$

Proof. Since ∂D is Jordan rectifiable by Lemma 2, the nonzero finite limit

$$f'(\zeta) = \lim_{z\to\zeta, z\in\bar{\mathbb{U}}} \frac{f(z)-f(\zeta)}{z-\zeta} \neq 0, \infty \tag{4.4}$$

exists for almost all $\zeta \in \mathbb{T}$, see [21, Theorem 6.8]. Assume that

$$0 < \beta_1 = |f'(e^{i\theta_1})| < |f'(e^{i\theta_2})| = \beta_2 < \infty \tag{4.5}$$

for $e^{i\theta_1}, e^{i\theta_2} \in l_f$. Note that (4.4), (4.5) allow us to apply the variation of Lemma 10.

Fix positive numbers k_1, k_2 such that

$$0 < k_1 < 1 < k_2 \quad \text{and} \quad k_1\beta_1^{-1} > k_2\beta_2^{-1}.$$

For fixed $\varphi > 0$ small enough consider the two point variation $\tilde{D}$ of D centered at $w_1 = f(e^{i\theta_1})$ and $w_2 = f(e^{i\theta_2})$ with inclinations φ and radii $\varepsilon_1 = k_1\varepsilon$, $\varepsilon_2 = k_2\varepsilon$, respectively.

Computing the change in the area by formula (3.32), we find

$$\operatorname{Area}\tilde{D} - \operatorname{Area}D = \frac{2\varphi - \sin 2\varphi}{2\sin^2\varphi}\varepsilon^2(k_1^2 - k_2^2) + o(\varepsilon^2).$$

Therefore,

$$\operatorname{Area}\tilde{D} < \operatorname{Area}D \tag{4.6}$$

for all $\varepsilon > 0$ small enough. Applying the variation (3.31) of Lemma 10, we get

$$m(\tilde{D},1) - m(D,1) = \left[\frac{\varphi(2+\varphi)}{6(1+\varphi)^2}\frac{k_1^2}{\beta_1^2} - \frac{\varphi(2-\varphi)}{6(1-\varphi)^2}\frac{k_2^2}{\beta_2^2}\right]\varepsilon^2 + o(\varepsilon^2) = \left[\frac{\varphi}{3}\left(\frac{k_1^2}{\beta_1^2} - \frac{k_2^2}{\beta_2^2}\right) + o(\varphi)\right]\varepsilon^2 + o(\varepsilon^2),$$

which implies that

$$R(\tilde{D},1) > R(D,1) \tag{4.7}$$

for all $\varepsilon > 0$ small enough if φ is chosen such that the expression in the brackets is positive.

Inequalities (4.6) and (4.7) contradict the monotonicity property of the function $A(\alpha)$. Thus $|f'(e^{i\theta})| = \beta$ with some $\beta > 0$ for almost all $e^{i\theta} \in l_f$.

To prove that $|f'(e^{i\theta})| < \beta$ for all $e^{i\theta} \in l_n$, we assume that $\beta = |f'(e^{i\theta_1})| < |f'(e^{i\theta_2})| = \beta_2$ with $e^{\theta_1} \in l_f$ and some $e^{\theta_2} \in l_n$. Then applying the two point variation as above we get inequalities (4.6), and (4.7), again contradicting the monotonicity property of $A(\alpha)$. Hence, $|f'(e^{i\theta})| \le \beta$ for all $e^{i\theta} \in l_n$, which combined with the strict monotonicity property of Corollary 2 leads to the strict inequality in (4.3). $\square$

Lemma 13 *If $1 < \alpha < 2$, then $l_n = \{e^{i\theta} : \theta_0 < \theta < 2\pi - \theta_0\}$ with some $0 < \theta_0 = \theta_0(\alpha) < \pi$; f' is continuous on $\bar{\mathbb{U}}$, and for all $z \in \overline{\mathbb{U}}$*

$$\beta_0 = |f'(-1)| \le |f'(z)| \le |f'(e^{i\theta})| = \beta, \tag{4.8}$$

where $e^{i\theta} \in \bar{l}_f$ and $0 < \beta_0 < \alpha < \beta$.

Proof. Since ∂D is Jordan rectifiable, f' belongs to the Hardy space H^1 [12] and therefore can be represented by the Poisson integral

$$f'(z) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta - t) f'(e^{it})\, dt \tag{4.9}$$

with boundary values defined a.e. on $\mathbb{T}$.

If $l_n = \emptyset$, then (4.9) combined with (4.3) shows that $|f'| = \beta$ identically on $\mathbb{U}$. Therefore, $f(z) = 1 + \beta z$ with $\beta = \alpha$, which is not in P_α since $\alpha > 1$. Therefore, $l_n \neq \emptyset$.

By Corollary 2, $|f'(e^{i\theta})|$ decreases in $\theta_0(\alpha) < \theta < \pi$. Besides, $f'(-1) := \beta_0 > 0$ since f is conformal at $z = -1$. Therefore inequalities (4.8) follow from (4.3) and representation (4.9). This implies that $\log|f'|$ is bounded harmonic in $\mathbb{U}$. Let h be a bounded harmonic function in $\mathbb{U}$ with boundary values $\log\beta$ on l_f and $\log|f'(e^{i\theta})|$ on l_n. Then $h - \log|f'|$ has nontangential limit zero a.e. on $\mathbb{T}$. Therefore, $h - \log|f'| \equiv 0$ in $\mathbb{U}$. Hence, $|f'| = \beta$ everywhere on l_f.

By the symmetry principle, f can be continued analytically through l_n and f' can be continued analytically through l_f. Thus we need to show only that f' is continuous at $e^{i\theta_0}$ and $e^{-i\theta_0}$.

Since $D \subset \mathbb{H}_r$ and $f(e^{i\theta_0}) = is_0 \in \partial\mathbb{H}_r$, it follows from the Julia-Wolff lemma, see [21, Proposition 4.13], that there exists a nonzero angular derivative $f'(e^{i\theta_0})$, which is finite by Lemma 11.

By the symmetry principle, f is analytic in the domain $\Delta^- = \{z \in \mathbb{U}_\varepsilon(e^{i\theta_0}) : \theta_0 < \arg z < \pi\}$, where $\varepsilon > 0$ and small enough. Since f has an angular derivative $f'(e^{i\theta_0})$ along any segment $I \subset \Delta^- \cap \mathbb{U}$ ending at $e^{i\theta_0}$, it follows [21, Proposition 4.9] that f has the same angular derivative along any nontangential path in Δ^- ending at $e^{i\theta}$. Therefore $f'(z) \to f'(e^{i\theta_0})$ along any such path.

Similarly, f' is analytic in $\Delta^+ = \{z \in \mathbb{U}_\varepsilon(e^{i\theta_0}) : 0 < \arg z < \theta_0\}$. By Proposition 4.7 [21] there is a finite angular limit $f'(e^{i\theta_0})$ along any segment $I \subset \Delta^+ \cap \mathbb{U}$ ending at $e^{i\theta_0}$. Proposition 4.9 [21] implies that f has angular derivative $f'(e^{i\theta_0})$ along any nontangential path in Δ^+ ending at $e^{i\theta_0}$. Again Proposition 4.7 [21] guarantees that f' has limit $f'(e^{i\theta_0})$ along any such path. Continuity of f' at $e^{i\theta_0}$ and, since f is symmetric with respect to $\mathbb{R}$, at $e^{-i\theta_0}$ is proved.

Corollary 2 and Lemma 12 show that $|f'(e^{i\theta})|$ takes its maximal value β on points $e^{i\theta} \in \bar{l}_f$ and its minimal value $\beta_0 > 0$ at the point $z = -1$. Since the extremal function in P_α is not linear for $1 < \alpha < 2$, the maximum principle for analytic functions implies $\beta_0 < |f'(z)| < \beta$ for all $z \in \mathbb{U}$, in particular $\beta_0 < \alpha = f'(0) < \beta$. □

Since f' is continuous and separated from 0 on $\bar{\mathbb{U}}$ and since $\arg(e^{i\theta} f'(e^{i\theta})) = \pi$ for $e^{i\theta} \in l_n$, the function $g(z) = \log f'(z)$ with the principal branch of the logarithm solves the mixed boundary value problem for bounded analytic functions in $\mathbb{U}$ with boundary conditions:

$$\begin{aligned} \Im g(e^{i\theta}) &= \pi - \theta \quad \text{if} \quad e^{i\theta} \in \bar{l}_n, \\ \Re g(e^{i\theta}) &= \beta \qquad\quad \text{if} \quad e^{i\theta} \in l_f. \end{aligned} \tag{4.10}$$

It is well known that, reducing this problem to the upper half-plane, its solution can be found by the Keldysh-Sedov formula [18]. Another method to solve problem (4.10) based on our knowledge of the boundary behavior of f and f' will be used in the next section.

5 Closed form for extremal functions and proof of Theorem 1

First we show that extremality of f implies not only univalence of f itself but also univalence of its derivative f' and univalence of the function $zf'(z)$.

Lemma 14 *If f is extremal in P_α with $1 < \alpha < 2$, then*

$$zf'(z) = \alpha\tau^{-1}p_\tau(z) \tag{5.1}$$

with some $0 < \tau < 1$, where $p_\tau(z)$ is the Pick function.

Proof. Let $\varphi(z) = zf'(z)$. Lemma 13 shows that φ is analytic in $\mathbb{U}$, continuous on $\bar{\mathbb{U}}$, and satisfies the inequality $|\varphi(z)| \le \beta$ with $\beta > \alpha$ defined in Lemma 12.

To describe the image $\varphi(\mathbb{T})$ of the boundary, we note that $\arg\,\varphi(e^{i\theta}) = \arg(e^{i\theta}f'(e^{i\theta})) = \pi$ for $\theta_0 \le \theta \le 2\pi - \theta_0$, where θ_0 is defined in Lemma 13. This together with Corollary 2 to Lemma 4 shows that φ maps the arc $\{e^{i\theta} : \theta_0 \le \theta \le \pi\}$ continuously and one-to-one onto the segment $I = \{w : \Im w = 0, -\beta \le \Re w \le -\beta_0\}$ such that $\varphi(e^{i\theta_0}) = -\beta$, $\varphi(-1) = -f'(-1) = -\beta_0$. By symmetry, φ maps the arc $\{e^{i\theta} : \pi \le \theta \le 2\pi - \theta_0\}$ onto I such that $\varphi(e^{i\theta_0}) = -\beta$.

Note that φ is analytic on l_f. If $\varphi'(e^{i\theta_1}) = 0$ for some $e^{i\theta_1} \in l_f$, analyticity implies that $\varphi(\mathbb{U}_\varepsilon(e^{i\theta_1})) \cap \bar{\mathbb{U}}$ with $\varepsilon > 0$ small enough covers some circular sector of opening $> \pi$ centered at $\varphi(e^{i\theta_1}) = \beta e^{i\psi}$ with some $\psi \in \mathbb{R}$. The latter contradicts the inequality (4.8): $|\varphi(z)| = |zf'(z)| \le \beta$. Thus $\varphi' \ne 0$ on l_f. This follows also from the Julia-Wolff lemma.

Since $|\varphi|$ is constant on l_f, for $e^{i\theta} \in l_f$ we get

$$\frac{\partial}{\partial\theta}\arg\,\varphi(e^{i\theta}) = -i\frac{\partial}{\partial\theta}\log\,\varphi(e^{i\theta}) = \frac{e^{i\theta}\varphi'(e^{i\theta})}{\varphi(e^{i\theta})} \ne 0.$$

Therefore $\arg\,\varphi(e^{i\theta})$ is monotone on l_f. Besides, $\arg\,\varphi(1) = 0$, $\arg\,\varphi(e^{i\theta_0}) = \pi$. Note that $\arg\,\varphi(e^{i\theta})$ defines the direction of outward normal of $D = f(\mathbb{U})$ at $f(e^{i\theta})$. Since D possesses Steiner symmetry, it follows that $0 \le \arg\,\varphi(e^{i\theta}) \le \pi$ for $0 \le \theta \le \theta_0$ and $-\pi \le \arg\,\varphi(e^{i\theta}) \le 0$ for $-\theta_0 \le \theta \le 0$. This analysis shows that φ maps l_f continuously and one-to-one onto the circle C_β punctured at $\zeta = -\beta$. Therefore φ maps $\mathbb{T}$ continuously and one-to-one in the sense of boundary correspondence onto the boundary of the domain $\Omega = \mathbb{U}_\beta \setminus [-\beta, -\beta_0]$. Now the argument principle yields that φ maps $\mathbb{U}$ conformally and univalently onto Ω.

Since $\varphi'(0) = f'(0) = \alpha$, we easily get (5.1). $\square$

One can easily check that (5.1) and (1.7) imply univalence of f'.

Proof of Theorem 1. Let f be extremal in P_α, $1 < \alpha < 2$. By (5.1),

$$f(z) = \alpha\tau^{-1} \int_{-1}^{z} t^{-1} p_\tau(t)\, dt, \tag{5.2}$$

where

$$p_\tau(z) = \frac{4\tau z}{(1 - z + \sqrt{(1-z)^2 + 4\tau z})^2} \tag{5.3}$$

is the Pick function. The normalization $f(0) = 1$ implies

$$1 = 4\alpha \int_0^1 \frac{dt}{(1 + t + \sqrt{(1+t)^2 - 4\tau t})^2}. \tag{5.4}$$

The denominator of the integrand in (5.4) decreases and therefore the integral itself, denoted by $I(\tau)$, increases in $0 < \tau < 1$. Since $I(0) = 1/8$ and $I(1) = 1/4$, for each $1 < \alpha < 2$ there exists a unique solution $\tau = \tau(\alpha)$ of (5.4) and therefore the extremal function f is defined by (5.2) with $\tau = \tau(\alpha)$. Evaluating the integral in (5.4) we obtain the equivalent equation (1.8).

To evaluate Area$\,(D)$, we apply a standard line integral formula and the fact that $\Im\,(\bar{w}\, dw) = 0$ on the nonfree boundary. We have

$$\text{Area}\,(D) = \frac{1}{2}\Im \int_{\partial D} \bar{w}\, dw = \frac{1}{2}\Im \int_{L_f} \bar{w}\, dw = \frac{1}{2}\Re \int_{-\theta_0}^{\theta_0} \overline{f(e^{i\theta})} e^{i\theta} f'(e^{i\theta})\, d\theta.$$

Since $|f'|^2 = \beta^2$ on l_f, we obtain

$$\text{Area}\,(D) = \frac{\beta^2}{2}\Re \int_{-\theta_0}^{\theta_0} \frac{f(e^{i\theta})}{e^{i\theta} f'(e^{i\theta})}\, d\theta = \frac{\beta^2}{2}\Re \int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta} f'(e^{i\theta})}\, d\theta = \frac{\beta^2}{2}\Im \int_{\mathbb{T}} \frac{f(z)}{z^2 f'(z)}\, dz$$

$$= \pi\beta^2 \Re \text{Res}\left[\frac{f(z)}{z^2 f'(z)}, 0\right] = \pi\beta^2 \Re\left\{ \left(\frac{f(z)}{f'(z)}\right)'\Bigg|_{z=0} \right\} = \pi\beta^2(1 - \alpha^{-2} f''(0)). \tag{5.5}$$

The second equality in this chain follows from the fact that $f(z)/(zf'(z))$ is purely imaginary on the nonfree arc l_n.

Computing the second derivative of the function (5.2), we get

$$f''(0) = 2\alpha(1 - \tau).$$

Therefore,

$$\text{Area}\,(D) = \pi\beta^2(1 - 2\alpha^{-1}(1 - \tau)).$$

Substituting $\beta = f'(1) = \alpha\tau^{-1}$, we get (1.5). $\square$

Proof of Corollary 1. Let $P_{\alpha,n} = \{f \in P : c_n(f) = \alpha\}$, $1 < \alpha < 2$. Then $P_{\alpha,n}$ is a compact convex subset of P. Therefore the arguments of Lemma 1 guarantee existence and uniqueness of a function f_n minimizing $D(f)$ in $P_{\alpha,n}$. The function $g(z) = f_n(e^{2\pi i/n}z)$ is in $P_{\alpha,n}$ and $D(g) = D(f)$. By uniqueness,

$$f_n(e^{2\pi i/n}z) = f_n(z).$$

This implies

$$c_m(f_n) = e^{2\pi im/n}c_m(f_n) \tag{5.6}$$

for all integer $m \geq 1$. (5.6) shows that $c_m(f_n) = 0$ if n does not divide m. Therefore $f_n(z) = f(z^n)$ with some $f \in P_\alpha$. Since $D(f_n) = n\,D(f)$, the corollary follows from Theorem 1. $\square$

6 Other minimal area problems. Progress and questions

We first describe the present status of Goodman's omitted area problem mentioned in the Introduction. (See [7, 15] for a general history of the problem.) We state here the generalized Goodman's problem : Find for any given $1/4 < R < \infty$, all functions f in the standard class S of univalent analytic in $\mathbb{U}$ functions that cover a minimal area $A(R)$ of the disc $\mathbb{U}_R$

$$A(R) = \min_{f\in S} \text{Area}(f(\mathbb{U}) \cap \mathbb{U}_R).$$

In the original setting of this problem $R = 1$.

All previously known properties of extremal functions were summarized in [7] and in Theorem 1 in [19]. The method of our paper allows us to prove all the previously known assertions and add some new results, which allow us to reduce Goodman's problem to a certain boundary value problem for analytic functions.

It is known that for $1/4 < R < R'$ with some $R' > 1$ every $f \in S$ minimizing $A(R)$ is unbounded.

It seems plausible that there is $1 < R_0 < \infty$ such that for $1/4 < R < R_0$ the extremal is unique and unbounded. For $R_0 < R < \infty$ the identity mapping is the unique extremal and in that case $A(R) = \pi$. For $R = R_0$ there are two extremal functions, one of which is $f(z) = z$.

If $f \in S$ is an unbounded extremal for $A(R)$, then $D = f(\mathbb{U})$ is circularly symmetric w.r.t. $\mathbb{R}_0$ (up to rotation about the origin) satisfying the following boundary conditions:

a) $f(1) = \infty$;

b) $\Im f(e^{i\theta}) = 0$ for $0 < |\theta| < \theta_1$;

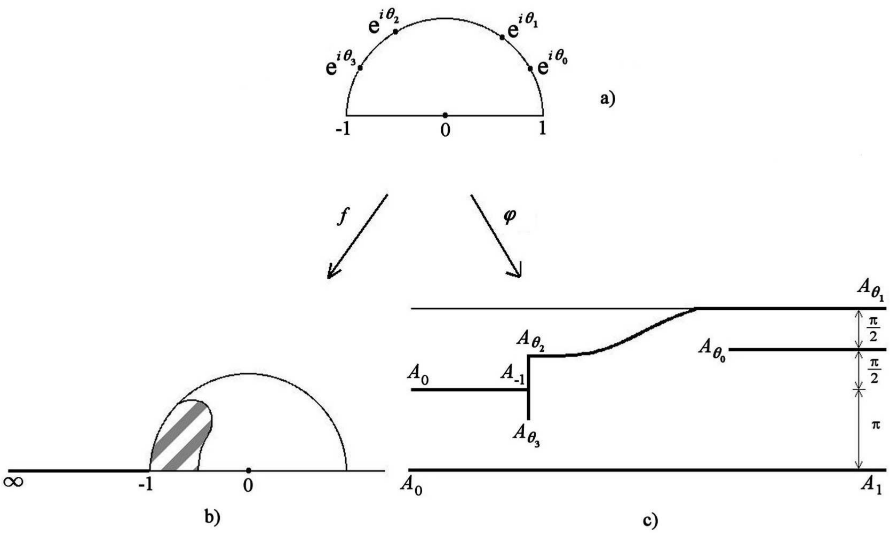

Figure 6: Extremal configurations for Goodman's problem

c) $|f(e^{i\theta})| = R$ for $\theta_1 < |\theta| < \theta_2$;

d) $|f'(e^{i\theta})| = \beta$ for $\theta_2 < \theta < 2\pi - \theta_2$;

e) f' has a non-zero continuous extension to $\mathbb{U} \cup \{e^{i\theta} : \theta_1 < \theta < 2\pi - \theta_1\}$ which is Hölder-continuous with exponent $1/2$.

Here $0 < \theta_1 < \theta_2 < \pi$ and $0 < \beta < 1$ are unknown parameters depending on R and f. Conditions a) - c) are well known. Conditions d) and e) were first proved by J. Lewis [19] for all θ such that $|f(e^{i\theta})| < R$ except the set $I = \{e^{i\theta} : \Im f(e^{i\theta}) = 0\}$. In [19] it was also shown that I might consist of at most a finite number of closed arcs. Using polarization w.r.t. circles centered at the origin we can easily show that $I = \{1\}$, which leads to d). Polarization allows us also to prove that $\arg f(e^{i\theta})$ strictly increases on some interval $\pi \le \theta \le \theta'$ with $\pi < \theta' < \theta_2$. Figure 6b shows the general shape of an extremal domain.

In addition to the above mentioned properties the method of this paper allows us to prove the following new ones:

f) $|f'(e^{i\theta})|$ strictly decreases in $\theta_1 < \theta < \theta_2$;

g) there is a θ_0, $0 < \theta_0 < \theta_1$ such that $|f'(e^{i\theta})|$ strictly decreases from $+\infty$ to some $\beta_1 > \beta$ and strictly increases from β_1 to $+\infty$ in $0 < \theta < \theta_0$ and $\theta_0 < \theta < \theta_1$, respectively.

To prove f) we use Lemma 5; g) follows from considerations of properties of level sets of $|f'|$ which are similar to arguments in [3, Section 5]. Summing up our knowledge about f, we can show that the function $\varphi(z) = \log zf'(z)$ maps the upper half-disc $\mathbb{U}^+$ univalently onto a simply connected domain depicted in Figure 6c such that the boundary points $0, 1, e^{i\theta_0}$, etc. correspond to the boundary points A_0, A_1, A_{θ_0}, etc.

We can also obtain similar results for *the hyperbolic metric - area problem* for closed

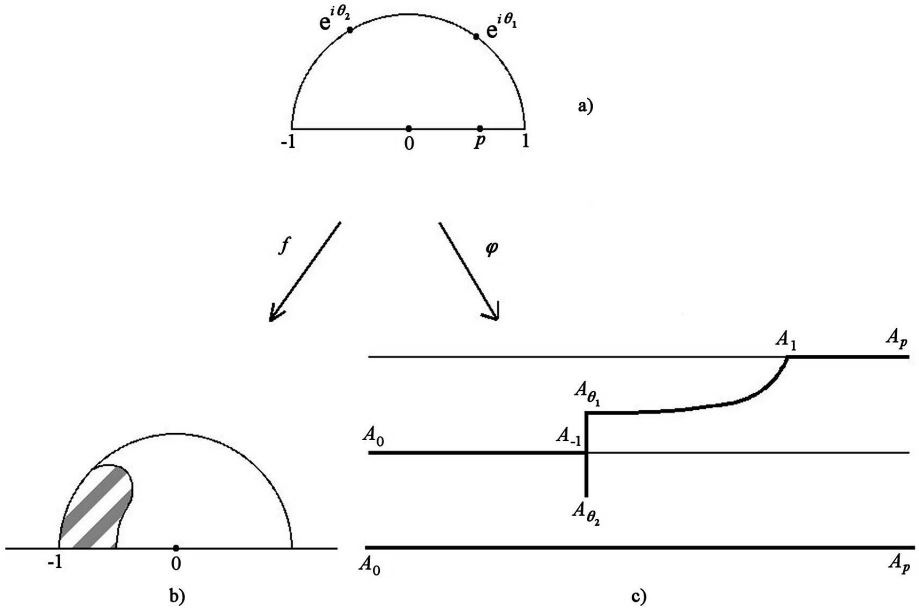

Figure 7: Extremal configurations for Solynin's problem

sets E on $\overline{\mathbb{C}}$ first studied by Solynin [26]. Let $\lambda_E(z)$ denote the hyperbolic metric of $D = \overline{\mathbb{C}} \setminus E$. It is well known that λ_E provides the maximal solution to the equation $\Delta \log \lambda_E = 4\lambda_E^2$ in D. For any given $R > 0$, let

$$\alpha = \frac{\text{Area}\,(E \cap \mathbb{U}_R(z))}{\pi R^2}$$

be the normalized area of E concentrated in the disc $\mathbb{U}_R(z)$. The problem is to find the sharp lower bound, $\mu(\alpha)$, for λ_E in terms of the normalized area α:

$$\lambda_E(z) \geq R^{-1}\mu(\alpha).$$

Taking $z = 0$, $R = 1$, it follows from the symmetrization results in [27, 25, 26] that any set E^* extremal for this problem is circularly symmetric w.r.t. the negative real axis (up to rotation about the origin), connected and touches the unit circle.

Just as for Goodman's problem, it seems possible that there is $0 < \alpha_0 < 1$ such that for $0 < \alpha < \alpha_0$ the extremal domain $D^* = \overline{\mathbb{C}} \setminus E^*$ is unique and unbounded as depicted in Figure 7b. For $\alpha_0 < \alpha < 1$, D^* is a disc $\mathbb{U}_r$ with $r = (1-\alpha)^{1/2}$. For $\alpha = \alpha_0$ there are two extremal domains, one of which is a disc.

Let f map $\mathbb{U}$ univalently onto D^* such that $f(0) = 0$, $f'(0) > 0$. If D^* is unbounded our method shows that $\varphi(z) = \log z f'(z)$ maps $\mathbb{U}^+$ univalently onto the domain depicted in Figure 7c with an obvious boundary correspondence.

As we already mentioned two minimal area problems with geometric constraints, similar to the problem of the present paper, were solved in [19] and [8]: Goodman's problems for starlike functions and for logarithmic area respectively. Our method can also be applied to these problems. In addition, we apply it in [10] to establish an isoperimetric inequality linking the area, diameter, and logarithmic capacity of a plane connected set.

A common feature of all problems where we have obtained an explicit resolution is that the corresponding function $\varphi(z) = \log zf'(z)$ maps $\mathbb{U}^+$ conformally onto a rectilinear polygon and therefore can be recovered by the Schwarz-Christoffel formula. In all these cases the problem under consideration reduces to a mixed boundary value problem for analytic functions which can be solved by the Keldysh-Sedov formula.

In contrast, the function φ corresponding to Goodman's or Solynin's problem maps $\mathbb{U}^+$ onto a domain having a curvilinear arc on the boundary. In this situation the problem can be also reduced to a boundary value problem for analytic function but with boundary conditions nonlinear with respect to the real and imaginary parts. As far as we know, very little is known about solutions of such problems. This explains why the corresponding minimal area problems remain open.

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