Computational Methods and Function Theory
Volume 1 (2001), No. 1, 81–88

A Note on the Hypergeometric Mean Value

Roger W. Barnard and Kendall C. Richards

Abstract. Recent efforts to obtain bounds for the complete elliptic integral

$$\frac{\pi}{2} \cdot {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right)$$

in terms of power means and other related means have precipitated the search for similar bounds for the more general ${}_2F_1(\alpha, \beta; \gamma; r)$. In an early paper, B. C. Carlson considered the approximation of the *hypergeometric mean values* $({}_2F_1(-a, b; b+c; r))^{1/a}$ in terms of *means of order* t, given by $M_t(s, r) := \{(1-s) + s(1-r)^t\}^{1/t}$. In this note, a refinement of one such approximation is established by first proving a general positivity result involving ${}_3F_2$.

Keywords. Hypergeometric function, generalized hypergeometric function, means of order t.

2000 MSC. 33C, 41A.

In the last few decades, there has been an intense renewed interest in the classical special functions, in particular the Gaussian hypergeometric function. This is evidenced by the almost 1000 papers listed just in the last three years in the Mathematical Reviews under the topic "hypergeometric functions." For an extensive bibliography and history see [1, 3, 4]. Hypergeometric functions, which have many of the classical special functions as special cases, have been found useful in resolving several current problems as noted in [5, 7, 8, 10, 11]. Given real numbers α, β, and γ with $\gamma \neq 0, -1, -2, \ldots$, the *Gaussian hypergeometric function* is defined by

$${}_2F_1(\alpha, \beta; \gamma; r) := \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{r^n}{n!}, \qquad |r| < 1.$$

Here $(\alpha)_0 = 1$ for $\alpha \neq 0$, and $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n = 1, 2, 3, \ldots$.

In [7], [8], R. Barnard, K. Pearce, and K. Richards proved very recently that the following inequalities are true for all $r \in [0, 1]$:

$$(1) \qquad \left(\frac{1+(r')^{3/2}}{2}\right)^{2/3} \leq {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) \leq \left(\frac{1+(r')^2}{2}\right)^{1/2}$$

Received September 24, 2001.

ISSN 1617-9447/\$ 2.50

where $r' = \sqrt{1-r^2}$. If, for $x, y, t > 0$, we use the notation

$$A_t(x,y) = ((x^t + y^t)/2)^{1/t}$$

for the power mean, we can write the lower and upper bounds as the power means $A_{3/2}(1,r')$ and $A_2(1,r')$, respectively.

One might seek a natural generalization of this inequality by replacing the parameter values $(-1/2, 1/2, 1)$ by a more general triple. B. C. Carlson [12] considered the approximation of the hypergeometric mean values in terms of means of order t. For $r, s, t > 0$, the *mean of order* t is given by

$$M_t(s,r) := \{(1-s) + s(1-r)^t\}^{1/t}$$

and the *hypergeometric mean of order* a is given by

$$M(a,b,c,r) := \{\,{}_2F_1(-a,b;b+c;r)\}^{1/a}$$

for $r \in [0,1]$, $a, b, c > 0$. Recall the following representation due to Euler (see [4], p. 65):

$$ {}_2F_1(-a,b;b+c;r) = \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} \int_0^1 u^{b-1}(1-u)^{c-1}(1-ur)^a\,du, \tag{2}$$

for $b, c > 0$, from which it follows that ${}_2F_1(-a,b;b+c;r) > 0$ for all $r \in (0,1)$. The fact that ${}_2F_1(-a,b;b+c;1)$ is finite for $b, c > 0$ follows from

$${}_2F_1(-a,b;b+c;1) = \frac{\Gamma(a+c)\Gamma(b+c)}{\Gamma(a+b+c)\Gamma(c)}$$

which is due to Gauss (see [4]). It is also helpful to note that for fixed $r, s \in (0,1)$ the function $t \mapsto M_t(s,r)$ is monotone (e.g., see [9], p. 17). The main result of this note is Theorem 1, which refines the following theorem of B. C. Carlson [12].

Theorem A (Carlson, 1965)**.** *If $a \in (0,1)$ and $b, c > 0$, then*

$$M_a\left(\frac{b}{b+c}, r\right) < M(a,b,c,r) \qquad \textit{for all } r \in (0,1). \tag{3}$$

Theorem B (Carlson, 1965)**.** *If $a > 1$ and $b, c > 0$, then*

$$M_a\left(\frac{b}{b+c}, r\right) > M(a,b,c,r) \qquad \textit{for all } r \in (0,1). \tag{4}$$

Sketch of Proof of Theorem A. Restricting our attention to the Gaussian hypergeometric function, Carlson's proof in [12] takes the following form: Note that $[M_a(u,r)]^a < [M_1(u,r)]^a$, since M_t is an increasing function of t and recall the integral representation for ${}_2F_1$ given in (2). Finally, for p and q positive integers it follows from $(\alpha)_\beta = \Gamma(\alpha+\beta)/\Gamma(\alpha)$ and properties of the Beta function B

(e.g., see [4]) that

$$\begin{aligned}\frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}\int_0^1 u^{b-1+p}(1-u)^{c-1+q}\,du &= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}B(b+p,c+q)\\ &= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}\frac{\Gamma(b+p)\Gamma(c+q)}{\Gamma(b+c+p+q)}\\ &= \frac{(b)_p(c)_q}{(b+c)_{p+q}}.\end{aligned}$$

Therefore

$$\begin{aligned}[\,M(a,b,c,r)\,]^a &= {}_2F_1(-a,b;b+c;r)\\ &= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}\int_0^1 u^{b-1}(1-u)^{c-1}[(1-u)+u(1-r)]^a\,du\\ &= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}\int_0^1 u^{b-1}(1-u)^{c-1}[M_1\,(u,r)]^a\,du\\ &> \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}\int_0^1 u^{b-1}(1-u)^{c-1}[M_a\,(u,r)]^a\,du\\ &= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}\int_0^1 u^{b-1}(1-u)^{c-1}[(1-u)+u(1-r)^a]\,du\\ &= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}\int_0^1 u^{b-1}(1-u)^{c-1+1}\,du\\ &\qquad +(1-r)^a\cdot\frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}\int_0^1 u^{b-1+1}(1-u)^{c-1}\,du\\ &= \frac{c+b(1-r)^a}{b+c} = \left[M_a\left(\frac{b}{b+c},r\right)\right]^a.\end{aligned}$$

■

In [12], Carlson uses an argument similar to that discussed above to prove Theorem B and that (3) holds for $-\infty < a < 0$ as well as $a = 0$ as a limiting case.

Since M_t is an increasing function of t, a natural question to ask is the following:

Question. *Given $a \in (0,1)$ and $b, c > 0$, are there values of $t > a$ such that*

$$M_a\left(\frac{b}{b+c},r\right) < M_t\left(\frac{b}{b+c},r\right) < M(a,b,c,r) \qquad \textit{for all } r \in (0,1)?$$

Applying Lemma 1, which is a general positivity result involving ${}_3F_2$ and is of independent interest (see [6, 7, 8]), we have obtained the following

Theorem 1. *Suppose $a \in (0,1)$ and $b, c > 0$. If $-\infty < t < (a+ab+c)/(1+b+c)$, then*

$$M_t\left(\frac{b}{b+c}, r\right) < M(a,b,c,r) \qquad \text{for all } r \in (0,1). \tag{5}$$

Remark. First note that $(a+ab+c)/(1+b+c) > a$. Also, since

$$M(a,b,c,r) - M_t\left(\frac{b}{b+c}, r\right) = \frac{bc}{2}\left[\frac{a+b+c-t(1+b+c)}{(b+c)^2(c+b+1)}\right] r^2 + O(r^3)$$

it follows that $t < (a+b+c)/(1+b+c)$ is a necessary condition for (5).

In order to prove Theorem 1, we will need the following

Lemma 1. *Suppose $\alpha, \beta, \gamma > 0$ and $1 > \lambda > \max\{\alpha\beta/\gamma, \alpha+\beta-\gamma\}$. Then*

$$(-\lambda)_n \cdot {}_3F_2(-n, \alpha, \beta; \gamma, 1+\lambda-n; 1) < 0 \qquad \text{for all } n \in \mathbb{N},$$

where ${}_3F_2$ is the generalized hypergeometric function given by

$${}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; r) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n(\alpha_3)_n}{(\beta_1)_n(\beta_2)_n}\frac{r^n}{n!}.$$

Proof of Lemma 1. Let $r \in (0,1)$ and define $h(r) := (1-r)^{\lambda}\,{}_2F_1(\alpha, \beta; \gamma; r)$. Thus

$$\begin{aligned} h(r) &= \sum_{n=0}^{\infty} \frac{(-\lambda)_n}{n!} r^n \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{r^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left\{\frac{(-\lambda)_{n-k}}{(1)_{n-k}}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}\right\} r^n \\ &= \sum_{n=0}^{\infty} \frac{(-\lambda)_n}{n!}\sum_{k=0}^{n}\left\{\frac{(-1)^k(-n)_k}{(-1)^k(1+\lambda-n)_k}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}\right\} r^n \end{aligned}$$

using $(\alpha)_{n-k} = (-1)^k(\alpha)_n/(1-\alpha-n)_k$. Thus

$$h(r) = \sum_{n=0}^{\infty} \frac{(-\lambda)_n}{n!}\,{}_3F_2(-n, \alpha, \beta; \gamma, 1+\lambda-n; 1) r^n.$$

It follows that

$$\begin{aligned}
h'(r) &= -\lambda(1-r)^{\lambda-1}\,{}_2F_1(\alpha,\beta;\gamma;r) + \frac{ab}{c}(1-r)^{\lambda}\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;r) \\
&= -(1-r)^{\lambda-1}\cdot\sum_{n=0}^{\infty}\left\{\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{r^n}{n!}\left(\lambda-(1-r)\frac{(\alpha+n)(\beta+n)}{(\gamma+n)}\right)\right\} \\
&= -(1-r)^{\lambda-1}\cdot\sum_{n=0}^{\infty}\left\{\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{r^n}{n!}\left(\lambda-\frac{(\alpha+n)(\beta+n)}{(\gamma+n)}\right)\right. \\
&\qquad\qquad \left. +\frac{(\alpha)_{n+1}(\beta)_{n+1}}{(\gamma)_{n+1}}\frac{r^{n+1}}{n!}\right\} \\
&= -(1-r)^{\lambda-1}\cdot\left[\sum_{n=0}^{\infty}\left\{\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{r^n}{n!}\left(\lambda-\frac{(\alpha+n)(\beta+n)}{(\gamma+n)}\right)\right\}\right. \\
&\qquad\qquad \left. +\sum_{n=0}^{\infty}\frac{n(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{r^n}{n!}\right] \\
&= -(1-r)^{\lambda-1}\cdot\sum_{n=0}^{\infty}\left\{\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{r^n}{n!}\left(\lambda+n-\frac{(\alpha+n)(\beta+n)}{(\gamma+n)}\right)\right\}.
\end{aligned}$$

The result now follows for $1>\lambda>\max\{\alpha\beta/\gamma,\alpha+\beta-\gamma\}$ by noting that $(1-r)^{\lambda-1}$ and

$$\sum_{n=0}^{\infty}\left\{\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{r^n}{n!}\left(\lambda+n-\frac{(\alpha+n)(\beta+n)}{(\gamma+n)}\right)\right\}$$

both have positive Maclaurin series coefficients. ■

Proof of Theorem 1. After (5) is proved for $0<t<(a+ab+c)/(1+b+c)$, the fact that (5) holds for $t<0$ (and $t=0$ as a limiting case, see [9, 12]) follows directly from the monotonicity of M_t. Let $a\in(0,1)$; $b,c>0$; $s=b/(b+c)$; and $0<t<(a+ab+c)/(1+b+c)$. Define

$$\begin{aligned}
f(r) &:= {}_2F_1(-a,b;b+c;r) = \sum_{n=0}^{\infty}A_nr^n, \\
g(r) &:= \left\{M_t\left(\frac{b}{b+c},r\right)\right\}^a = \sum_{n=0}^{\infty}B_nr^n.
\end{aligned}$$

It follows that $B_0 = A_0 = 1$ and $B_1 = A_1 = -as$. Now suppose that $B_k \leq A_k$ for all $k = 1, ..., n$. The logarithmic derivative of g becomes

$$\frac{g'(r)}{g(r)} = \frac{-as}{(1-s)(1-r)^{1-t} + s(1-r)}$$

and thus

$$\left\{\sum_{n=0}^{\infty}(n+1)B_{n+1}r^n\right\}\left\{(1-s)(1-r)^{1-t} + s(1-r)\right\} = -as\sum_{n=0}^{\infty}B_n r^n. \tag{6}$$

Using $(1-r)^{1-t} = \sum_{n=0}^{\infty}(t-1)_n r^n/n!$ and the Cauchy product, we find that

$$\begin{aligned}
(n+1)B_{n+1} &= B_n[s(n-a) - n(1-s)(t-1)] \\
&\quad -(1-s)\sum_{k=0}^{n-2}(k+1)B_{k+1}\frac{(t-1)_{n-k}}{(1)_{n-k}} \\
&\leq A_n[s(n-a) - n(1-s)(t-1)] \\
&\quad -(1-s)\sum_{k=0}^{n-2}(k+1)A_{k+1}\frac{(t-1)_{n-k}}{(1)_{n-k}} \\
&= A_n s(n-a) + (1-s)(n+1)A_{n+1} \\
&\quad -(1-s)\sum_{k=0}^{n}(k+1)A_{k+1}\frac{(t-1)_{n-k}}{(1)_{n-k}} \\
&= (n+1)A_{n+1} + A_n s(n-a) - s(n+1)A_{n+1} + \frac{as(1-s)}{n!}\cdot \\
&\quad \cdot(t-1)_n\,{}_3F_2(-n, 1-a, b+1; b+c+1, 2-t-n; 1) \\
&= (n+1)A_{n+1} + \frac{as(1-s)}{n!}\cdot \\
&\quad \cdot\left\{(t-1)_n\,{}_3F_2(-n, 1-a, b+1; b+c+1, 2-t-n; 1)\right. \\
&\quad \left. -\frac{(1-a)_n(b)_n}{(b+c+1)_n}\right\} \\
&\leq (n+1)A_{n+1}
\end{aligned}$$

using Lemma 1 with $\lambda = 1-t$, $\alpha = 1-a$, $\beta = b+1$, and $\gamma = b+c+1$, since $0 < t < (a+ab+c)/(1+b+c)$ implies $1 > \lambda > (1-a)(b+1)/(b+c+1) = \alpha\beta/\gamma > \alpha+\beta-\gamma$. ■

It should be noted that if $t = (a+b+c)/(1+b+c)$, then (5) does not hold for $b > c$. In the case that $b = c$ we make the following

Conjecture. *Suppose $a \in (0,1)$, $b > 0$, and $a+b > 1/2$. If $t < (a+2b)/(1+2b)$, then*

$$M_t\left(\frac{1}{2}, r\right) < M(a,b,b,r) \tag{7}$$

holds for all $r \in (0,1)$. The bound $(a+2b)/(1+2b)$ is sharp.

Additional motivation for this work and a proof of this conjecture in the case that $a = b = 1/2$ can be found in [7, pp. 693–694]. In particular, we have shown

Theorem 2 (Barnard, Pearce, Richards, 2000)**.** *If $t < 3/4$, then*

$$M_t\left(\frac{1}{2}, r\right) < M\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, r\right) \qquad \text{for all } r \in (0,1). \tag{8}$$

The bound $3/4$ is sharp.

In view of Theorem B and the fact that M_t is an increasing function of t, the following problem remains for future investigation:

Problem. Given $a, b, c > 0$, identify conditions on a, b, c, and the sharp value $\phi(a,b,c)$ such that if $t > \phi(a,b,c)$, then

$$M_t\left(\frac{b}{b+c}, r\right) > M(a,b,c,r) \qquad \text{for all } r \in (0,1).$$

It should also be noted that in the special case that $a = b = c = 1/2$, H. Alzer [2] has conjectured that if $\phi_0 = \ln(2)/(2\ln(\pi/2)) \approx 0.767$ and $t > \phi_0$, then

$$M_t\left(\frac{1}{2}, r\right) > M\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, r\right) \qquad \text{for all } r \in (0,1). \tag{9}$$

The bound ϕ_0 is sharp. (Note that (1) implies that (9) holds for all $t \geq 2$.)

Acknowledgment. The authors wish to thank the referees for their suggestions that have considerably enhanced this paper.

References

1. G. Almkvist and B. Berndt, Gauss, Landen, Ramanujan, the arithmetic-geometric mean, ellipses, π, and the Ladies Diary, *Amer. Math. Monthly*, **95** (1988), 585–608.
2. Horst Alzer, personal correspondence, 1998.
3. G. D. Anderson, M. K. Vamanamurthy, and M. K. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, Wiley, New York, 1997.
4. G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, 1999.
5. R. Askey, *Orthogonal Polynomials and Special Functions*, SIAM, Philadelphia, 1975.
6. R. Askey, G. Gasper, and M. Ismail, A positive sum from summability theory, *J. Approx. Theory* **13** (1975), 413–420.
7. R. W. Barnard, K. Pearce, and K. C. Richards, An inequality involving the generalized hypergeometric function and the arc length of an ellipse, *SIAM J. Math. Anal. 31* (2000), 693–699.

8. R. W. Barnard, K. Pearce, and K. C. Richards, A monotonicity property involving ${}_3F_2$ and comparisons of classical approximations of elliptical arc length, *SIAM J. Math. Anal.* **32** (2000), 403–419.
9. E. F. Beckenbach and R. Bellman, *Inequalities*, fourth printing, Springer-Verlag, Berlin, 1983.
10. J. Borwein and P. Borwein, *Pi and the AGM*, Wiley, New York, 1987.
11. J. Borwein and P. Borwein, Inequalities for compound mean iterations with logarithmic asymptotes, *J. Math. Anal. Appl. 177* (1993), 572–582.
12. B. C. Carlson, Some inequalities for hypergeometric functions, *Proc. Amer. Math. Soc.* **16** (1966), 32–39.

Roger W. Barnard E-MAIL: `barnard@math.ttu.edu`
ADDRESS: *Department of Mathematics and Statistics, Texas Tech University, Lubbock, Texas 79409-1042, U.S.A.*

Kendall C. Richards E-MAIL: `richardk@southwestern.edu`
ADDRESS: *Department of Mathematics, Southwestern University, Georgetown, Texas 78626, U.S.A.*