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Univalency of weighted integral transforms of certain functions

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Abstract

For $\beta < 1$ and $\gamma \geqslant 0$, let $\mathscr{P}_\gamma(\beta)$ denote the class of all normalized analytic functions f in the unit disc such that

$$\mathrm{Re}\left\{e^{i\eta}\left((1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-\beta\right)\right\}>0, \quad z \in \Delta,$$

for some $\eta \in \mathbb{R}$. For $f \in \mathscr{P}_\gamma(\beta)$, we consider the integral transform

$$V_\lambda(f)(z)=\int_0^1 \lambda(t)\frac{f(tz)}{t}\,dt,$$

where $\lambda(t)$ is a real-valued nonnegative weight function so that $\int_0^1 \lambda(t)\,dt = 1$. The main aim of this paper is to find conditions such that $V_\lambda(f) \in \mathscr{P}_1(\alpha)$ whenever $f \in \mathscr{P}_\gamma(\beta)$ for $\gamma \geqslant 1$. We also obtain conditions such that $V_\lambda(f) \in \mathscr{P}_1(\beta')$ whenever $f \in \mathscr{P}_0(\beta)$ for various choices of $\lambda(t)$. As a useful consequence, we find conditions for certain hypergeometric functions to be univalent.

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1. Introduction and main results

Let $\mathscr{A}$ denote the class of analytic functions f in the unit disk $\Delta=\{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0)=0=f'(0)-1$. For $\beta < 1$ and $\gamma \geqslant 0$, let $\mathscr{P}_\gamma(\beta)$ denote the class of all analytic functions f in $\mathscr{A}$ such that

$$\operatorname{Re}\left\{e^{i\eta}\left((1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-\beta\right)\right\}>0, \quad z \in \Delta,$$

for some $\eta \in \mathbb{R}$. Set $\mathscr{P}_0(\beta) =: \mathscr{P}(\beta)$ and denote $\mathscr{P}(0)$ simply by $\mathscr{P}$. For $0 \leqslant \beta < 1$, functions in $\mathscr{P}_1(\beta)$ are known to be univalent in Δ. For a general reference for the special classes of univalent functions we refer to [6,8].

An immense number of papers have been published concerning operators on linear combinations of $f(z)/z$ and $f'(z)$ primarily because these combinations and their operators appear naturally in criteria for univalency, subordination and positivity results on classes of functions. By allowing the parameters η, γ and β to vary in our class $\mathscr{P}_\gamma(\beta)$ we encompass a large number of the previous results. In particular, by incorporating general properties of convolutions and several formulae involving hypergeometric functions we are able to overcome many of the previous obstacles to obtain the general sought-after criteria.

The Gaussian hypergeometric function $F(a, b; c; z)$ defined by the series

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)}z^n \quad (a,b,c \in \mathbb{C},\ c \notin \{0,-1,-2,\ldots\})$$

is analytic in the unit disc Δ. Here $(a, 0) = 1$ for $a \neq 0$, and (a, n) is the shifted factorial function

$$(a,n)=a(a+1)(a+2)\cdots(a+n-1)$$

for $n = 1, 2, \ldots$. This function has two different representations. If $\operatorname{Re} c > \operatorname{Re} b > 0$, then we have the Euler integral representation

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt. \tag{1.1}$$

Moreover, if $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$, $\operatorname{Re}(c + 1) > \operatorname{Re}(a + b)$, then we have the following representation [3,9]:

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}\int_0^1 \lambda_1(t)\frac{1}{1-tz}\,dt, \tag{1.2}$$

where $\lambda_1(t)=t^{b-1}(1-t)^{c-a-b}F(c-a,1-a;c-a-b+1;1-t)$. This representation has been quite useful. For $f \in \mathscr{A}$, we define the integral transform

$$V_\lambda(f)(z)=\int_0^1 \lambda(t)\frac{f(tz)}{t}\,dt. \tag{1.3}$$

Here $\lambda(t)$ is any real-valued nonnegative weight function normalized so that $\int_0^1 \lambda(t)\,dt=1$. This operator contains some of well-known operators such as Libera, Bernardi and Komatu as its special cases. This operator has been studied by a number of authors for various choices of $\lambda(t)$ [3,12,14,16,15,7].

We recall the following results which give conditions on β' and $\lambda(t)$ so that:

(i) $f \in \mathscr{P}_1(\beta) \Rightarrow V_\lambda(f) \in \mathscr{P}_1(\beta')$ (see [7])
(ii) $f \in \mathscr{P}_\gamma(\beta), \gamma \in [0, 1] \Rightarrow V_\lambda(f) \in \mathscr{P}_\gamma(\beta')$ (see [9])
(iii) $f \in \mathscr{P}_\gamma(\beta), \gamma \in (0, 1] \Rightarrow f \in \mathscr{P}(\beta')$ (see [9]).

The special case concerning $\mathscr{P}_1(\beta)$ mapping into $\mathscr{P}_1(\beta')$, for $\lambda(t) = (1 + c)t^c$, has been considered in [7]. We are interested in the following problem because of its applications to the classical problem of determining when $F(a, b; c; z)$ is univalent.

Problem 1.4. Find conditions so that

(i) $f \in \mathscr{P}_\gamma(\beta), \gamma \in [1, \infty) \Rightarrow V_\lambda(f) \in \mathscr{P}_1(\beta')$
(ii) $f \in \mathscr{P}_0(\beta) \Rightarrow V_\lambda(f) \in \mathscr{P}_1(\beta')$.

We note that (ii) is particularly useful because it provides functions $f \in \mathscr{A}$ with their Maclaurin coefficients $a_n = \mathrm{O}(1)$ such that the corresponding operator $V_\lambda(f)$ is univalent and a solution which has not been available in the literature with an approach which previously had been difficult to use. Our first result answers Problem 1.4(i) and the estimate here is sharp.

Theorem 1.5. *Let $\alpha < 1$, $\gamma \geqslant 1$ be given, and define $\beta = \beta(\alpha, \gamma)$ by*

$$\beta = 1 - \frac{1-\alpha}{2}\left\{1 - \frac{1}{\gamma}\int_0^1 \frac{\lambda(t)}{1+t}\,\mathrm{d}t + \left(\frac{1}{\gamma} - 1\right)\int_0^1 \lambda(t)\left(\int_0^1 \frac{\mathrm{d}u}{1+tu^\gamma}\right)\mathrm{d}t\right\}^{-1}.$$

If $f \in \mathscr{P}_\gamma(\beta)$ then $V_\lambda f \in \mathscr{P}_1(\alpha)$. The value of β is sharp.

The proof of the main theorems will be supplied in Section 3. The case $\gamma = 1$ of Theorem 1.5 gives

Corollary 1.6 (*Fournier and Ruscheweyh [7]*). *Let $\alpha < 1$, and define $\beta(\alpha) := \beta(\alpha, 1) < 1$ by*

$$\beta(\alpha) = 1 - (1-\alpha)\left[2\left(1 - \int_0^1 \frac{\lambda(t)}{1+t}\,\mathrm{d}t\right)\right]^{-1}.$$

If $f \in \mathscr{P}_1(\beta(\alpha))$ then $V_\lambda f \in \mathscr{P}_1(\alpha)$.

The counterpart of Theorem 1.5 for $0 < \gamma < 1$ is still open. On the other hand, for $\lambda(t) = (1 + c)t^c$, we have

Theorem 1.7 (*Barnard et al. [4, Theorem 4.4]*). *Let $-1 < c \leqslant 0$ and*

$$G(z) = \frac{1+c}{z^c}\int_0^z t^{c-1} f(t)\,\mathrm{d}t.$$

Suppose that $\mathrm{Re}\{\mathrm{e}^{\mathrm{i}\eta}((f(z)/z) - \beta)\} > 0$, $z \in \Delta$. *Then we have*

$$\mathrm{Re}\{\mathrm{e}^{\mathrm{i}\eta}(G'(z) - \gamma)\} > 0,$$

where $\gamma = 1 - 2(1-\beta)(1-\beta')$ *with*

$$\beta' = \frac{1+c}{2} - c\int_0^1 \frac{\mathrm{d}t}{1+t^{1/(1+c)}} = \frac{1+c}{2} - cF(1, 1+c; 2+c; -1).$$

In particular, *we have*

(1) $\mathrm{Re}\, \mathrm{e}^{\mathrm{i}\eta}\left(\frac{f(z)}{z} - \frac{1-2\beta'}{2(1-\beta')}\right) > 0$ *implies* $\mathrm{Re}\, \mathrm{e}^{\mathrm{i}\eta} G'(z) > 0$,
(2) $\mathrm{Re}\, \mathrm{e}^{\mathrm{i}\eta}\left(\frac{f(z)}{z} - \frac{1}{2}\right) > 0$ *implies* $\mathrm{Re}\, \mathrm{e}^{\mathrm{i}\eta}(G'(z) - \beta') > 0$.

This result motivates Problem 1.4(ii). In this paper, we also solve Problem 1.4(ii) for certain special choices of $\lambda(t)$ and one of the results generalizes Theorem 1.7.

Theorem 1.8. *Let* $0 < a \leqslant 1$, $b < c - a \leqslant 1/a$ *and* $H = H_{a,b;c}$ *be the convolution operator defined by*

$$H_{a,b;c}(f(z)) := [H_{a,b;c}(f)](z) = zF(a, b; c; z) * f(z)$$

(*see also* (2.3)). *Suppose that* $f \in \mathscr{P}(\beta)$. *Then we have* $H(f) \in \mathscr{P}_1(\gamma)$ *where* $\gamma = 1 - 2(1-\beta)(1-\beta')$ *with*

$$\beta' = (1-a)F(a, b; c; -1) + aF(a+1, b; c; -1). \tag{1.9}$$

The result is sharp. In particular, *we have*
(1) $\mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\eta}\left(\frac{f(z)}{z} - \frac{1-2\beta'}{2(1-\beta')}\right)\right\} > 0$ *implies* $\mathrm{Re}\{\mathrm{e}^{\mathrm{i}\eta} H(f)'(z)\} > 0$,
(2) $\mathrm{Re}\{\mathrm{e}^{\mathrm{i}\eta}(\frac{f(z)}{z} - \frac{1}{2})\} > 0$ *implies* $\mathrm{Re}\{\mathrm{e}^{\mathrm{i}\eta}(H(f)'(z) - \beta')\} > 0$.

At this place it is interesting to recall the often studied class of functions f such that $\mathrm{Re}(f(z)/z) > \frac{1}{2}$, $z \in \Delta$. There exist sufficient conditions for a function f to be in this class. For example if f is in

$$\mathscr{U} = \left\{ f \in \mathscr{A} : \left| \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 \right| < 1,\ z \in \Delta \right\},$$

then [10]

$$\mathrm{Re}\left(\frac{f(z)}{z}\right) > \frac{2}{4 + |f''(0)|}, \quad z \in \Delta.$$

In particular, if f is in $\mathscr{U}_2 = \{f \in \mathscr{U} : f''(0) = 0\}$, then $\mathrm{Re}(f(z)/z) > \frac{1}{2}$, $z \in \Delta$. Further, if $f \in \mathscr{A}$ and $\mathrm{Re}(zf'(z)/f(z)) > \frac{1}{2}$, then it is well known that $\mathrm{Re}(f(z)/z) > \frac{1}{2}$, $z \in \Delta$. Note that every convex function is not necessarily in $\mathscr{P}_1(0)$. Moreover, if f belongs to the class $\mathscr{R}(\lambda)$ defined by

$$\mathscr{R}(\lambda) = \{f \in \mathscr{A} : |f'(z) - 1| < \lambda,\ z \in \Delta\},$$

then $\mathscr{R}(\lambda) \nsubseteq \mathscr{S}^*$ if $\lambda > \frac{2}{\sqrt{5}}$ (see [17]).

Corollary 1.10. *Let* $0 < b \leqslant c - 1 \leqslant 1$ *and* $h(z) = H_{1,b;c}(f(z))$ *be defined by the Carlson–Shaffer operator given by*

$$h(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{f(tz)}{t}\, \mathrm{d}t.$$

Suppose that $f \in \mathscr{P}(\beta)$. *Then we have* $h \in \mathscr{P}_1(\gamma)$, *where* $\gamma = 1 - 2(1-\beta)(1-\beta')$ *with* $\beta' = F(2, b; c; -1)$. *The constant* γ *is sharp.*

Proof. The case $b < c - 1 \leqslant 1$ follows from Theorem 1.8 if we choose $a = 1$. It remains to show the result for $c = b + 1$. However, this case is actually Theorem 1.7. □

Suppose

$$G(z) := G_f(a, b; z) = \left(\sum_{n=1}^{\infty} \frac{(1+a)(1+b)}{(n+a)(n+b)} z^n \right) * f(z). \tag{1.11}$$

Then, we have [14]

$$G_f(a, b; z) = \int_0^1 \lambda(t) \frac{f(tz)}{t}\, \mathrm{d}t \tag{1.12}$$

where

$$\lambda(t) = \begin{cases} (a+1)(b+1) \left(\dfrac{t^a(1-t^{b-a})}{b-a} \right) & \text{for } b \neq a,\ a > -1,\ b > -1, \\ (a+1)^2 t^a \log(1/t) & \text{for } b = a,\ a > -1. \end{cases}$$

This operator has been introduced in [11] and has been studied by a number of authors [15,3,2]. Because of the symmetry, we may assume $b > a$ if $b \neq a$.

Theorem 1.13. *Let* $-1 < a < 0$, $b > a$ *and* $f \in \mathscr{P}(\beta)$. *Then* G *defined by* (1.11) *is in* $\mathscr{P}_1(\gamma)$, *where* $\gamma = 1 - 2(1-\beta)(1-\beta')$ *with*

$$\beta' = \begin{cases} \dfrac{(a+1)(b+1)}{b-a} \int_0^1 \dfrac{(bt^b - at^a)}{1+t}\, \mathrm{d}t & \textit{for } b \neq a, \\ (1+a)^2 \int_0^1 \dfrac{(1 + a\log t)t^a}{1+t}\, \mathrm{d}t & \textit{for } b = a. \end{cases}$$

As $\beta' > 0$, the choice $f(z) = z/(1-z)$ (and hence $\beta = \frac{1}{2}$) shows that the second order polylogarithm given by

$$\sum_{n=1}^{\infty} \frac{(1+a)(1+b)}{(n+a)(n+b)} z^n$$

belongs to $\mathscr{P}_1(\beta')$. Moreover, this function is convex in Δ.

Our next result concerns the following convolution operator discussed by Ponnusamy and Sabapathy [16]:

$$F_{a,p}(z) = \left(\sum_{n=1}^{\infty} \frac{(1+a)^p}{(n+a)^p} z^n \right) * f(z). \tag{1.14}$$

Theorem 1.15. *Let* $-1 < a \leqslant 0$, $p > 1$ *and* $f(z) \in \mathscr{P}(\beta)$. *Then* $F_{a,p}$ *defined by* (1.14) *belongs to* $\mathscr{P}_1(\gamma)$, *where* $\gamma = 1 - 2(1-\beta)(1-\beta')$ *with*

$$\beta' = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 (\log 1/t)^{p-2} \frac{t^a}{1+t} (p - 1 - a \log 1/t) \, \mathrm{d}t.$$

As before, under the hypotheses of Theorem 1.15, the pth order polylogarithm given by

$$\sum_{n=1}^{\infty} \frac{(1+a)^p}{(n+a)^p} z^n$$

belongs to $\mathscr{P}_1(\beta')$. The convexity of this function is clear at least when p is a positive integer.

2. Main Lemmas

Lemma 2.1 (*Ponnusamy [14]*). *Let* $\beta_1 < 1$ *and* $\beta_2 < 1$. *Then, for* p, q *analytic in* Δ *with* $p(0) = q(0) = 1$, *the conditions* $\operatorname{Re} p(z) > \beta_1$ *and* $\operatorname{Re} \mathrm{e}^{\mathrm{i}\eta}(q(z) - \beta_1) > 0$ *imply* $\operatorname{Re} \mathrm{e}^{\mathrm{i}\eta}((p * q)(z) - \delta) > 0$, *where* $1 - \delta = 2(1-\beta_1)(1-\beta_2)$.

Lemma 2.2 (*Balasubramanian et al. and Kim and Rønning [3,9]*). *Suppose that* $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$, *and* $\operatorname{Re}(c+1) > \operatorname{Re}(a+b)$. *Then, for* $f \in \mathscr{A}$, *we have*

$$H_{a,b;c}(f(z)) := [H_{a,b;c}(f)](z) = zF(a,b;c;z) * f(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} \, \mathrm{d}t, \tag{2.3}$$

where

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} t^{b-1} (1-t)^{c-a-b} F\left(\begin{matrix} c-a, \ 1-a \\ c-a-b+1 \end{matrix} ; 1-t \right).$$

The integral representation (2.3) has been obtained in [3,9] and the problem concerning its geometric properties has been discussed for example in [3,9,13,2].

3. Proofs of Theorems 1.5, 1.8, 1.13 and 1.15

3.1. Proof of Theorem 1.5

Define

$$(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)=p_\gamma(z).$$

Then, $zp_\gamma \in \mathscr{P}(\beta)$ is equivalent to $f \in \mathscr{P}_\gamma(\beta)$. Assume first that $\gamma \neq 0$. Then it is a simple exercise to see that

$$f'(z)=p_\gamma(z) * F(2,1/\gamma;1+1/\gamma;z). \tag{3.2}$$

In the case $\gamma=0$, we just write

$$f'(z)=(zp_\gamma(z))'=p_\gamma(z) * \frac{1}{(1-z)^2}$$

which is actually the limiting case of (3.2):

$$\lim_{\gamma\to 0} F(2,1/\gamma;1+1/\gamma;z)=\lim_{\gamma\to 0}\left(\sum_{n=0}^{\infty}\frac{n+1}{n\gamma+1}z^n\right)=\frac{1}{(1-z)^2}.$$

Now, we let $F(z)=V_\lambda(f)(z)$, where $V_\lambda(f)$ is defined by (1.3). Then for $\gamma \neq 0$ we can write

$$\begin{aligned}F'(z)&=f'(z) * \int_0^1 \frac{\lambda(t)}{1-tz}\,\mathrm{d}t\\ &=p_\gamma(z) * F(2,1/\gamma;1+1/\gamma;z) * \int_0^1 \frac{\lambda(t)}{1-tz}\,\mathrm{d}t\\ &=p_\gamma(z) * \left[\int_0^1 \lambda(t)F(2,1/\gamma;1+1/\gamma;tz)\,\mathrm{d}t\right].\end{aligned}$$

Again in the case $\gamma=0$, we just write

$$F'(z)=p_\gamma(z) * \left[\int_0^1 \frac{\lambda(t)}{(1-tz)^2}\,\mathrm{d}t\right]$$

which is in fact the same as taking the limit $\gamma \to 0$ in the previous expression that is valid for $\gamma \neq 0$. Since $f \in \mathscr{P}_\gamma(\beta)$, it follows that $\mathrm{Re}\{e^{i\eta}(p_\gamma(z)-\beta)\}>0$ for some $\eta \in \mathbb{R}$. Now, for each $\gamma \geqslant 0$, we first claim that

$$\mathrm{Re}\left[\int_0^1 \lambda(t)F(2,1/\gamma;1+1/\gamma;tz)\,\mathrm{d}t\right]>1-\frac{1-\alpha}{2(1-\beta)}, \quad z \in \Delta, \tag{3.3}$$

which, by Lemma 2.1, implies that $F \in \mathscr{P}_1(\alpha)$ which will complete the proof. Therefore, it suffices to verify the inequality (3.3). Using the identity (which can be checked by comparing the coefficients of z^n on both sides)

$$F(2,b;c;z)=(c-1)F(1,b;c-1;z)-(c-2)F(1,b;c;z),$$

it follows that

$$F(2,1/\gamma;1+1/\gamma;z)=\frac{1}{\gamma(1-z)}+\left(1-\frac{1}{\gamma}\right)\int_0^1\frac{\mathrm{d}u}{1-zu^\gamma}.$$

In view of this,

$$\int_0^1\lambda(t)F(2,1/\gamma;1+1/\gamma;tz)\,\mathrm{d}t=\frac{1}{\gamma}\int_0^1\frac{\lambda(t)}{1-tz}\,\mathrm{d}t+\left(1-\frac{1}{\gamma}\right)\int_0^1\lambda(t)\left(\int_0^1\frac{\mathrm{d}u}{1-tzu^\gamma}\right)\mathrm{d}t.$$

Therefore, for $\gamma\geqslant 1$, one has

$$\operatorname{Re}\left[\int_0^1\lambda(t)F(2,1/\gamma;1+1/\gamma;tz)\,\mathrm{d}t\right]>\frac{1}{\gamma}\int_0^1\frac{\lambda(t)}{1+t}\,\mathrm{d}t+\left(1-\frac{1}{\gamma}\right)\int_0^1\lambda(t)\left(\int_0^1\frac{\mathrm{d}u}{1+tu^\gamma}\right)\mathrm{d}t.$$

The stated condition on β shows that the right-hand side of the last expression is

$$1-\frac{1-\alpha}{2(1-\beta)}.$$

To prove the sharpness, let $f\in\mathscr{P}_\gamma(\beta)$ be the function determined by

$$(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)=\beta+(1-\beta)\frac{1+z}{1-z}.$$

Using a series expansion we see that we can write

$$f(z)=z+2(1-\beta)\sum_{n=2}^{\infty}\frac{1}{n\gamma+1-\gamma}z^n.$$

Then we can write

$$F(z)=V_\lambda(f)(z)=z+2(1-\beta)\sum_{n=2}^{\infty}\frac{\mu_n}{n\gamma+1-\gamma}z^n,$$

where $\mu_n=\int_0^1\lambda(t)t^{n-1}\,\mathrm{d}t$. From the given value of β in Theorem 1.5, it follows that

$$\begin{aligned}\frac{1}{1-\beta}&=\frac{2}{1-\alpha}\left[1-\frac{1}{\gamma}\int_0^1\frac{\lambda(t)}{1+t}\,\mathrm{d}t+\left(\frac{1}{\gamma}-1\right)\int_0^1\lambda(t)\left(\int_0^1\frac{\mathrm{d}u}{1+tu^\gamma}\right)\mathrm{d}t\right]\\&=\frac{2}{1-\alpha}\left[1+\int_0^1\lambda(t)\left\{-\frac{1}{\gamma}\frac{1}{1+t}+\left(\frac{1}{\gamma}-1\right)\int_0^1\frac{\mathrm{d}u}{1+tu^\gamma}\right\}\mathrm{d}t\right]\\&=\frac{2}{1-\alpha}\int_0^1\lambda(t)\left\{\sum_{n=2}^{\infty}(-1)^{n-1}t^{n-1}\left(-\frac{1}{\gamma}+\left(\frac{1}{\gamma}-1\right)\frac{1}{n\gamma+1-\gamma}\right)\right\}\mathrm{d}t\\&=-\frac{2}{1-\alpha}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}n\mu_n}{n\gamma+1-\gamma}.\end{aligned}$$

Finally, we see that

$$F'(z) = 1 + 2(1-\beta)\sum_{n=2}^{\infty}\frac{n\mu_n}{n\gamma+1-\gamma}z^{n-1}$$

which for $z=-1$ takes the value

$$1 + 2(1-\beta)\sum_{n=2}^{\infty}\frac{(-1)^{n-1}n\mu_n}{n\gamma+1-\gamma} = 1 + 2(1-\beta)\left(-\frac{1-\alpha}{2(1-\beta)}\right) = \alpha.$$

This shows that the result is sharp. □

3.4. Proof of Theorem 1.8

We have $H(z) = f(z) * zF(a,b;c;z)$ and therefore,

$$H'(z) = \frac{f(z)}{z} * M(z),$$

where, by the derivative formula for the hypergeometric function,

$$M(z) = F(a,b;c;z) + z\frac{ab}{c}F(a+1,b+1;c+1;z).$$

Using the contiguous relation [1, Eq. (2.5.5)] (which may be also verified by comparing the coefficient of z^n on both sides)

$$cF(a+1,b;c;z) = bzF(a+1,b+1;c+1;z) + cF(a,b;c;z),$$

we see that

$$M(z) = (1-a)F(a,b;c;z) + aF(a+1,b;c;z).$$

Now, we assume that $c > a+b$. In view of the integral representation (1.2), it is a simple exercise to see that

$$M(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b)}\int_0^1 t^{b-1}(1-t)^{c-a-b-1}\frac{N(t)}{1-tz}\,\mathrm{d}t,$$

where

$$N(t) = \frac{(1-a)}{c-a-b}(1-t)F\left(\begin{matrix} c-a,\ 1-a \\ c-a-b+1 \end{matrix};1-t\right) + F\left(\begin{matrix} c-a-1,\ -a \\ c-a-b \end{matrix};1-t\right)$$

$$= \frac{1-a}{c-a-b}\sum_{n=0}^{\infty}\frac{(c-a,n)(1-a,n)}{(c-a-b+1,n)(1,n)}(1-t)^{n+1} + \sum_{n=0}^{\infty}\frac{(c-a-1,n)(-a,n)}{(c-a-b,n)(1,n)}(1-t)^{n}$$

$$= 1+\sum_{n=0}^{\infty}\left[(1-a)\frac{(c-a,n)(1-a,n)}{(c-a-b,n+1)(1,n)} - a\frac{(c-a-1,n+1)(1-a,n)}{(c-a-b,n+1)(1,n+1)}\right](1-t)^{n+1}$$

$$= 1+\sum_{n=0}^{\infty}R(n)(1-t)^{n+1}.$$

Here

$$R(n) = (1-a)\frac{(c-a,n)(1-a,n)}{(c-a-b,n+1)(1,n)} - a(c-a-1)\frac{(c-a,n)(1-a,n)}{(c-a-b,n+1)(1,n+1)}$$

$$= \frac{(c-a,n)(1-a,n)}{(c-a-b,n+1)(1,n+1)}[(1-a)n+1-a(c-a)]$$

which is clearly nonnegative for all $n \geqslant 0$ as $a \in (0,1]$ and $b < c-a \leqslant 1/a$. In particular, $N(t) \geqslant 0$ for all $t \in [0,1]$.

Now, for $|z| \leqslant r$ $(0<r<1)$ and $0 \leqslant t \leqslant 1$, we have

$$\left|\frac{1}{1-tz} - \frac{1}{1-r^2t^2}\right| \leqslant \frac{rt}{1-r^2t^2},$$

which gives

$$\frac{1}{1+rt} \leqslant \mathrm{Re}\,\frac{1}{1-tz} \leqslant \frac{1}{1-rt}.$$

Thus, for $b < c-a \leqslant 1$ and $0 < a \leqslant 1$, we have $\mathrm{Re}\, M(z) > M(-1)$ and the estimate here is clearly sharp as we see below. Finally, we assume that $f \in \mathscr{P}(\beta)$. Now, we choose

$$\frac{f(z)}{z} = \phi(z) = 1+2(1-\beta)\frac{z}{1-z} \quad \text{and} \quad M(z) = \psi(z) = 1+2(1-\beta')\frac{z}{1-z}.$$

Further, with $\gamma = 1-2(1-\beta)(1-\beta')$, we have

$$\frac{f(z)}{z} * M(z) = (\phi * \psi)(z) = 1+4(1-\beta)(1-\beta')\frac{z}{1-z} = 1+2(1-\gamma)\frac{z}{1-z}.$$

Finally, we note that

$$\mathrm{Re}\,\frac{f(z)}{z} > \beta = \phi(-1), \quad \mathrm{Re}\, M(z) > \beta' = M(-1), \quad \mathrm{Re}\left(\frac{f(z)}{z} * M(z)\right) > \gamma$$

and the desired conclusion follows. □

3.5. Proof of Theorem 1.13

Case (*i*): Let $a \in (-1, 0)$, $b > a$ and G be defined by (1.11). It is a simple exercise to see that

$$G'(z) = M(z) * \frac{f(z)}{z}$$

where

$$M(z) = \frac{(a+1)(b+1)}{b-a}\left[-a\sum_{n=0}^{\infty}\frac{z^n}{n+a+1} + b\sum_{n=0}^{\infty}\frac{z^n}{n+b+1}\right].$$

In terms of the hypergeometric function, we can rewrite the last expression as

$$M(z) = \frac{1}{b-a}\left[-a(b+1)F(1, a+1; a+2; z) + b(a+1)F(1, b+1; b+2; z)\right].$$

By a direct integration (also by Euler's integral representation (1.1)), it is easy to see that

$$M(z) = \frac{(a+1)(b+1)}{b-a}\left[\int_0^1 \frac{1}{1-tz}(bt^b - at^a)\,\mathrm{d}t\right].$$

As $bt^b - at^a$ is positive for $a < 0$, $a < b$ and $t \in (0, 1)$, the method of proof of Theorem 1.8 shows that

$$\operatorname{Re} M(z) > M(-1) \quad \text{for } |z| < 1.$$

Case (*ii*): For $a = b$, we have

$$G(z) = \left(\sum_{n=1}^{\infty}\frac{(1+a)^2}{(n+a)^2}z^n\right) * f(z)$$

and, by a simple calculation, we see that

$$G'(z) = M(z) * \frac{f(z)}{z}, \quad M(z) = (1+a)^2\int_0^1 \frac{(1 + a\log t)t^a}{1-tz}\,\mathrm{d}t.$$

This representation may also be obtained by taking the limit $b \to a$ in the previous case $b \neq a$. The remaining part of the proof is as in the previous theorem. □

3.6. Proof of Theorem 1.15

For $p > 0$ and $a > -1$, we recall the integral representation [16]

$$F_{a,p}(z) = \frac{(1+a)^p}{\Gamma(p)}\int_0^1 (\log 1/t)^{p-1}t^{a-1}f(tz)\,\mathrm{d}t. \tag{3.7}$$

As in the proof of Theorems 1.8 and 1.13, we can easily obtain that

$$F'_{a,p}(z) = \frac{f(z)}{z} * M(z), \quad M(z) = \sum_{n=1}^{\infty}\frac{(1+a)^p(n+a-a)}{(n+a)^p}z^{n-1}.$$

A computation and the representation (3.7) gives that

$$M(z)=\frac{(1+a)^p}{\Gamma(p)}\int_0^1 (\log 1/t)^{p-2}\frac{t^a}{1-tz}(p-1-a\log 1/t)\,\mathrm{d}t.$$

For $p>1$ and $-1<a\leqslant 0$, we conclude that $\operatorname{Re} M(z)>M(-1)$ and the conclusion is clear as in the proof of Theorem 1.8. □

4. Univalency of hypergeometric functions

Choose

$$f(z)=z\left[\frac{1+(1-2\alpha)z}{1-z}\right] \quad \text{with } \alpha=\frac{1-2\beta'}{2(1-\beta')}$$

so that $2(1-\alpha)=1/(1-\beta')$. Then, it is a simple exercise to see that

$$H(z)=f(z)*zF(a,b;c;z)=z+2(1-\alpha)z(F(a,b;c;z)-1).$$

From the proof of Theorem 1.8, we observe that $M(-1)=\beta'>0$ because of the fact that the integrand in the integral representation of $M(-1)$ is positive on (0, 1). Thus we have the following result concerning the combination of z and $zF(a,b;c;z)$:

Corollary 4.1. *Let* $0<a\leqslant 1$, $b<c-a\leqslant 1/a$ *and* β' *be given by* (1.9). *Then*

$$H(z)=z\left(1-\frac{1}{1-\beta'}\right)+\frac{1}{1-\beta'}zF(a,b;c;z)$$

satisfies the condition $\operatorname{Re} H'(z)>\beta'$ *in* Δ. *In particular*, *H is univalent in* Δ.

The question of univalency of a convex combination of two functions is dealt with in [18]. In fact, Trimble [18] showed that if f is a normalized convex function, then the function F defined by

$$F(z)=\lambda z+(1-\lambda)f(z)$$

is starlike of order $\beta=(1-3\lambda)/(2(2+\lambda))$ provided $0\leqslant\lambda\leqslant\frac{1}{3}$. Related problems were considered later by a number of authors (e.g., [5,19]), by imposing an additional condition on f. Corollary 4.1 is a special situation of Trimble. However, because $\beta'>0$, the choice $f(z)=z/(1-z)$ in Corollary 4.1 gives

Corollary 4.2. *Let* $0<a\leqslant 1$, $b<c-a\leqslant 1/a$ *and* β' *be given by* (1.9). *Then for* $H(z)=zF(a,b;c;z)$, *we have* $\operatorname{Re} H'(z)>\beta'$ *in* Δ. *In particular*, *H is univalent in* Δ.

Although it seems difficult to find a neat form of the value of β' in general, for particular cases one can state it in a precise form. To make this point more clear, for example, for $c=a-b+1$, we recall that [1, Corollary 3.1.2]

$$F(a,b;a-b+1;-1)=2^{-a}\sqrt{\pi}\frac{\Gamma(1+a-b)}{\Gamma((a+1)/2)\Gamma(1+a/2-b)} \tag{4.3}$$

and the contiguous relation

$$c(1-z)F(a,b;c;z)=cF(a-1,b;c;z)+(b-c)zF(a,b;c+1;z),$$

which for $z=-1$ and $c=a-b$ gives

$$F(a,b;a-b;-1)=\frac{1}{2}F(a-1,b;a-b;-1)+\frac{a-2b}{2(a-b)}F(a,b;a-b+1;-1).$$

By (4.3), a simplification immediately gives

$$\begin{aligned}&F(a,b;a-b;-1)\\&\quad=2^{-a}\sqrt{\pi}\Gamma(a-b)\left\{\frac{1}{\Gamma(a/2-b)\Gamma((a+1)/2)}+\frac{a}{2\Gamma((a+1)2-b)\Gamma(a/2+1)}\right\}\end{aligned}$$

so that

$$\begin{aligned}\beta'&=(1-a)F(a,b;a-b+1;-1)+aF(a+1,b;a-b+1;-1)\\&=2^{-a}\sqrt{\pi}\Gamma(1+a-b)\left\{\frac{1-a/2}{\Gamma(1+a/2-b)\Gamma((a+1)/2)}+\frac{a/2}{\Gamma((a+1)/2-b)\Gamma(a/2+1)}\right\}.\end{aligned}$$

Corollary 4.4. *For $b>0$, $b+1<c\leqslant 2$, $H(z)=zF(1,b;c;z)$ is close-to-convex with respect to z (and hence, univalent in Δ).*

This corollary follows from Theorem 1.8. Indeed, according to Theorem 1.8 (with $f(z)=z/(1-z)$ and $a=1$), we have $\operatorname{Re} H'(z)>\beta'=F(2,b;c;-1)>0$, $z\in\Delta$, where $H(z)=zF(1,b;c;z)$. To provide an estimate to β', we use Euler's transformation (see [1, Theorem 2.2.5]) and see that

$$F(2,b;c;z)=(1-z)^{c-2-b}F(c-2,c-b;c;z).$$

The Euler integral representation (1.1) gives

$$F(2,b;c;z)=(1-z)^{c-2-b}\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{c-b-1}(1-t)^{b-1}(1-tz)^{-c+2}\,\mathrm{d}t$$

for $c>c-b>0$. Since this function is analytic in the cut plane $\mathbb{C}\backslash[1,\infty]$, we can write

$$F(2,b;c;-1)=\frac{2^{c-b-2}\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{c-b-1}(1-t)^{b-1}(1+t)^{-c+2}\,\mathrm{d}t. \tag{4.5}$$

Using a well-known result that [1, p. 50]

$$\int_0^1 t^{x-1}(1-t)^{y-1}\frac{1}{(t+p)^{x+y}}\,\mathrm{d}t=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)(1+p)^x p^y}$$

($x>0$, $y>0$, $p>0$) it follows by using $(1+t)^{-c}(1+t)^2>(1+t)^{-c}$ that

$$\int_0^1 t^{c-b-1}(1-t)^{b-1}(1+t)^{-c+2}\,\mathrm{d}t>\frac{\Gamma(c-b)\Gamma(b)}{\Gamma(c)}\frac{1}{2^{c-b}}.$$

In particular, by (4.5), this crude estimate implies that $\beta'=F(2,b;c;-1)>\frac{1}{4}$.

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