Financial Time Series

Lecture 5: More Volatility Models

Package Note: We use `fGarch` to estimate most volatility models, but will discuss the package `rugarch` later, which can be used to estimate GRACH-M, IGARCH, and EGARCH models.

The GARCH-M model

$$r_t = \mu + c\sigma_t^2 + a_t, \quad a_t = \sigma_t \epsilon_t, \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

where c is referred to as the risk premium, which is expected to be positive.

Example: A GARCH(1,1)-M model for the monthly excess returns of S&P 500 index from January 1926 to December 1991. The fitted model is

$$r_t = 4.22 \times 10^{-3} + 0.561\sigma_t^2 + a_t, \sigma_t^2 = 0.814 \times 10^{-5} + 0.122 a_{t-1}^2 + .854\sigma_{t-1}^2.$$

Standard error of the estimated risk premium is 0.896 so that the estimate is not statistically significant at the usual 5% level.

R demonstration

```
> source("garchM.R")
> sp5=scan(file="sp500.txt")
> m1=garchM(sp5)
Maximized log-likehood:  1269.053

Coefficient(s):
         Estimate  Std. Error  t value   Pr(>|t|)
mu    4.22469e-03 2.40670e-03  1.75539  0.0791929 .
gamma 5.61297e-01 8.96194e-01  0.62631  0.5311105
omega 8.13623e-05 2.92094e-05  2.78548  0.0053449 **
alpha 1.21976e-01 2.21373e-02  5.50995 3.5893e-08 ***
beta  8.54361e-01 2.22261e-02 38.43945 < 2.22e-16 ***
```

Remarks: The R script `garchM` is relatively slow. The intensive compoutation is due to using a recursive loop to evaluate the likelihood function.

The EGARCH model

The idea (concept) of EGARCH model is useful. In practice, it is easier to use the TGARCH model.

Asymmetry in responses to past positive and negative returns:

$$g(\epsilon_t) = \theta\epsilon_t + \gamma[|\epsilon_t| - E(|\epsilon_t|)],$$

with $E[g(\epsilon_t)] = 0$.

To see asymmetry of $g(\epsilon_t)$, rewrite it as

$$g(\epsilon_t) = \begin{cases} (\theta + \gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t \geq 0, \\ (\theta - \gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t < 0. \end{cases}$$

An EGARCH(m, s) model:

$$a_t = \sigma_t\epsilon_t, \; \ln(\sigma_t^2) = \alpha_0 + \frac{1 + \beta_1 B + \cdots + \beta_{s-1}B^{s-1}}{1 - \alpha_1 B - \cdots - \alpha_m B^m} g(\epsilon_{t-1}).$$

Some features of EGARCH models:

- uses log trans. to relax the positiveness constraint
- asymmetric responses

Consider an EGARCH(1,1) model

$$a_t = \sigma_t\epsilon_t, \qquad (1 - \alpha B)\ln(\sigma_t^2) = (1 - \alpha)\alpha_0 + g(\epsilon_{t-1}),$$

Under normality, $E(|\epsilon_t|) = \sqrt{2/\pi}$ and the model becomes

$$(1-\alpha B)\ln(\sigma_t^2) = \begin{cases} \alpha_* + (\theta+\gamma)\epsilon_{t-1} & \text{if } \epsilon_{t-1} \geq 0, \\ \alpha_* + (\theta-\gamma)\epsilon_{t-1} & \text{if } \epsilon_{t-1} < 0 \end{cases}$$

where $\alpha_* = (1-\alpha)\alpha_0 - \sqrt{\frac{2}{\pi}}\gamma$.

This is a nonlinear fun. similar to that of the threshold AR model of Tong (1978, 1990).

Specifically, we have

$$\sigma_t^2 = \sigma_{t-1}^{2\alpha}\exp(\alpha_*)\begin{cases} \exp[(\theta+\gamma)\frac{a_{t-1}}{\sqrt{\sigma_{t-1}^2}}] & \text{if } a_{t-1} \geq 0, \\ \exp[(\theta-\gamma)\frac{a_{t-1}}{\sqrt{\sigma_{t-1}^2}}] & \text{if } a_{t-1} < 0. \end{cases}$$

The coefs $(\theta+\gamma)$ & $(\theta-\gamma)$ show the asymmetry in response to positive and negative a_{t-1}. The model is, therefore, nonlinear if $\theta \neq 0$. Thus, θ is referred to as the leverage parameter.

Focus on the function $g(\epsilon_{t-1})$. The leverage parameter θ shows the effect of the sign of a_{t-1} whereas γ denotes the magnitude effect.

See Nelson (1991) for an exmaple of EGARCH model.

Another example: Monthly log returns of IBM stock from January 1926 to December 1997 for 864 observations.

For textbook, an AR(1)-EGARCH(1,1) is obtained (RATS program):

$$\begin{aligned} r_t &= 0.0105 + 0.092 r_{t-1} + a_t, \quad a_t = \sigma_t\epsilon_t \\ \ln(\sigma_t^2) &= -5.496 + \frac{g(\epsilon_{t-1})}{1-.856B}, \\ g(\epsilon_{t-1}) &= -.0795\epsilon_{t-1} + .2647[|\epsilon_{t-1}| - \sqrt{2/\pi}], \end{aligned}$$

Model checking:

For $\tilde{a}_t$: $Q(10) = 6.31(0.71)$ and $Q(20) = 21.4(0.32)$

For $\tilde{a}_t^2$: $Q(10) = 4.13(0.90)$ and $Q(20) = 15.93(0.66)$

Discussion:

Using $\sqrt{2/\pi} \approx 0.7979 \approx 0.8$, we obtain

$$\ln(\sigma_t^2) = -1.0 + 0.856 \ln(\sigma_{t-1}^2) + \begin{cases} 0.1852\epsilon_{t-1} & \text{if } \epsilon_{t-1} \geq 0 \\ -0.3442\epsilon_{t-1} & \text{if } \epsilon_{t-1} < 0. \end{cases}$$

Taking anti-log transformation, we have

$$\sigma_t^2 = \sigma_{t-1}^{2\times 0.856} e^{-1.001} \times \begin{cases} e^{0.1852\epsilon_{t-1}} & \text{if } \epsilon_{t-1} \geq 0 \\ e^{-0.3442\epsilon_{t-1}} & \text{if } \epsilon_{t-1} < 0. \end{cases}$$

For a standardized shock with magnitude 2, (i.e. two standard deviations), we have

$$\frac{\sigma_t^2(\epsilon_{t-1} = -2)}{\sigma_t^2(\epsilon_{t-1} = 2)} = \frac{\exp[-0.3442 \times (-2)]}{\exp(0.1852 \times 2)} = e^{0.318} = 1.374.$$

Therefore, the impact of a negative shock of size two-standard deviations is about 37.4% higher than that of a positive shock of the same size.

Forecasting: some recursive formula available

Another parameterization of EGARCH models

$$\ln(\sigma_t^2) = \alpha_0 + \alpha_1 \frac{|a_{t-1}| + \gamma_1 a_{t-1}}{\sigma_{t-1}} + \beta_1 \ln(\sigma_{t-1}^2),$$

where γ_1 denotes the leverage effect.

Below, I re-analyze the IBM log returns by extending the data to December 2009. The sample size is 1008.

The fitted model is

$$
\begin{aligned}
r_t &= 0.012 + a_t, \quad a_t = \sigma_t \epsilon_t \\
\ln(\sigma_t^2) &= -0.611 + \frac{0.231|a_{t-1}| - 0.250 a_{t-1}}{\sigma_{t-1}} + 0.92 \ln(\sigma_{t-1}^2).
\end{aligned}
$$

Since EGARCH and TGARCH (below) share similar objective and the latter is easier to estimate. We shall use the TGARCH model.

The Threshold GARCH (TGARCH) or GJR Model A TGARCH(s, m) or GJR(s, m) model is defined as

$$
r_t = \mu_t + a_t, \quad a_t = \sigma_t \epsilon_t \sigma_t^2 = \alpha_0 + \sum_{i=1}^{s} (\alpha_i + \gamma_i N_{t-i}) a_{t-i}^2 + \sum_{j=1}^{m} \beta_j \sigma_{t-j}^2,
$$

where N_{t-i} is an indicator variable such that

$$
N_{t-i} = \begin{cases} 1 & \text{if } a_{t-i} < 0, \\ 0 & \text{otherwise.} \end{cases}
$$

One expects γ_i to be positive so that prior negative returns have higher impact on the volatility.

The Asymmetric Power ARCH (APARCH) Model. This model was introduced by Ding, Engle and Granger (1993) as a general class of volatility models. The basic form is

$$
\begin{aligned}
r_t &= \mu_t + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim D(0, 1) \\
\sigma_t^\delta &= \omega + \sum_{i=1}^{s} \alpha_i (|a_{t-i}| - \gamma_i a_{t-i})^\delta + \sum_{j=1}^{m} \beta_j \sigma_{t-j}^\delta
\end{aligned}
$$

where δ is a non-negative real number. In particular, $\delta = 2$ gives rise to the TGARCH model and $\delta = 0$ corresponds to using $\log(\sigma_t)$.

Theoretically, one can use any power δ to obtain a model. In practice, two things deserve further consideration. First, δ will also affect the specification of the mean equation, i.e., model for μ_t. Second, it is hard to interpret δ, except for some special values such as 0, 1, 2.
In **R**, one can fix the value of δ a priori using the subcommand include.delta=F, delta = 2.
Here I pre-fix $\delta = 2$. Thus, we can use APARCH model to estimate TGARCH model. Consider the percentage log returns of monthly IBM stock from 1926 to 2009.

R demonstration

```
> da=read.table("m-ibm2609.txt",header=T)
> head(da)
      date       ibm
1 19260130 -0.010381
.....
6 19260630  0.068493
> ibm=log(da$ibm+1)*100
> m1=garchFit(~aparch(1,1),data=ibm,trace=F,delta=2,include.delta=F)
> summary(m1)
Title:
 GARCH Modelling
Call:
 garchFit(formula = ~aparch(1, 1), data = ibm, delta = 2, include.delta = F,
    trace = F)

Mean and Variance Equation:
 data ~ aparch(1, 1)
 [data = ibm]

Conditional Distribution:  norm

Coefficient(s):
     mu    omega   alpha1   gamma1    beta1
1.18659  4.33663  0.10767  0.22732  0.79468

Std. Errors:  based on Hessian
```

```
Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
mu       1.18659     0.20019    5.927 3.08e-09 ***
omega    4.33663     1.34161    3.232  0.00123 **
alpha1   0.10767     0.02548    4.225 2.39e-05 ***
gamma1   0.22732     0.10018    2.269  0.02326 *
beta1    0.79468     0.04554   17.449  < 2e-16 ***
---
Log Likelihood:
 -3329.177    normalized:  -3.302755

Standardised Residuals Tests:
                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  67.07416  2.775558e-15
 Shapiro-Wilk Test  R    W      0.9870142 8.597234e-08
 Ljung-Box Test     R    Q(10)  16.90603  0.07646942
 Ljung-Box Test     R    Q(15)  24.19033  0.06193099
 Ljung-Box Test     R    Q(20)  31.89097  0.04447407
 Ljung-Box Test     R^2  Q(10)  4.591691  0.9167342
 Ljung-Box Test     R^2  Q(15)  11.98464  0.6801912
 Ljung-Box Test     R^2  Q(20)  14.79531  0.7879979
 LM Arch Test       R    TR^2   7.162971  0.8466584

Information Criterion Statistics:
     AIC      BIC      SIC     HQIC
6.615430 6.639814 6.615381 6.624694
> plot(m1)  <= shows normal distribution is not a good fit.
>
> m1=garchFit(~aparch(1,1),data=ibm,trace=F,delta=2,include.delta=F,cond.dist="std")
> summary(m1)
Title:
 GARCH Modelling
Call:
 garchFit(formula = ~aparch(1, 1), data = ibm, delta = 2, cond.dist = "std",
    include.delta = F, trace = F)

Mean and Variance Equation:
 data ~ aparch(1, 1)
 [data = ibm]

Conditional Distribution: std

Coefficient(s):
     mu    omega   alpha1   gamma1    beta1    shape
1.20476  3.98975  0.10468  0.22366  0.80711  6.67329
```

```
Std. Errors: based on Hessian

Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
mu       1.20476     0.18715    6.437 1.22e-10 ***
omega    3.98975     1.45331    2.745 0.006046 **
alpha1   0.10468     0.02793    3.747 0.000179 ***
gamma1   0.22366     0.11595    1.929 0.053738 .
beta1    0.80711     0.04825   16.727  < 2e-16 ***
shape    6.67329     1.32779    5.026 5.01e-07 ***
---
Log Likelihood:
 -3310.21    normalized:  -3.283938

Standardised Residuals Tests:
                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  67.82336  1.887379e-15
 Shapiro-Wilk Test  R    W      0.9869698 8.212564e-08
 Ljung-Box Test     R    Q(10)  16.91352  0.07629962
 Ljung-Box Test     R    Q(15)  24.08691  0.06363224
 Ljung-Box Test     R    Q(20)  31.75305  0.04600187
 Ljung-Box Test     R^2  Q(10)  4.553248  0.9189583
 Ljung-Box Test     R^2  Q(15)  11.66891  0.7038973
 Ljung-Box Test     R^2  Q(20)  14.18533  0.8209764
 LM Arch Test       R    TR^2   6.771675  0.872326

Information Criterion Statistics:
     AIC      BIC      SIC     HQIC
6.579782 6.609042 6.579711 6.590898
> plot(m1)
Make a plot selection (or 0 to exit):

 1:   Time Series
 2:   Conditional SD
 3:   Series with 2 Conditional SD Superimposed
 4:   ACF of Observations
 5:   ACF of Squared Observations
 6:   Cross Correlation
 7:   Residuals
 8:   Conditional SDs
 9:   Standardized Residuals
10:   ACF of Standardized Residuals
11:   ACF of Squared Standardized Residuals
12:   Cross Correlation between r^2 and r
13:   QQ-Plot of Standardized Residuals
```

```
Selection: 13

### The APARCH model with delta = 2 can also be fitted by using the
         subcommand ``leverage=T'' as below
mm <- garchFit(~garch(1,1),data=ibm, trace=F, leverage=T)
```

For the percentage log returns of IBM stock from 1926 to 2009, the fitted GJR model is

$$\begin{aligned} r_t &= 1.20 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t^*_{6.67} \\ \sigma_t^2 &= 3.99 + 0.105(|a_{t-1}| - 0.224 a_{t-1})^2 + .807\sigma_{t-1}^2, \end{aligned}$$

where all estimates are significant, and model checking indicates that the fitted model is adequate.

Note that, we can obtain the model for the log returns as

$$\begin{aligned} r_t &= 0.012 + a_t, \quad a_t = \sigma_t \epsilon_t \\ \sigma_t^2 &= 3.99 \times 10^{-4} + 0.105(|a_{t-1}| - 0.224 a_{t-1})^2 + .807\sigma_{t-1}^2. \end{aligned}$$

The sample variance of the IBM log returns is about 0.005 and the empirical 2.5% percentile of the data is about -0.130. If we use these two quantities for σ_{t-1}^2 and a_{t-1}, respectively, then we have

$$\begin{aligned} \frac{\sigma_t^2(-)}{\sigma_t^2(+)} &= \frac{0.0004 + 0.105(0.130 + 0.224 \times 0.130)^2 + 0.807 \times 0.005}{0.0004 + 0.105(0.130 - 0.224 \times 0.130)^2 + 0.807 \times 0.005} \\ &= 1.849. \end{aligned}$$

In this particular case, the negative prior return has about 85% higher impact on the conditional variance.

Stochastic volatility model

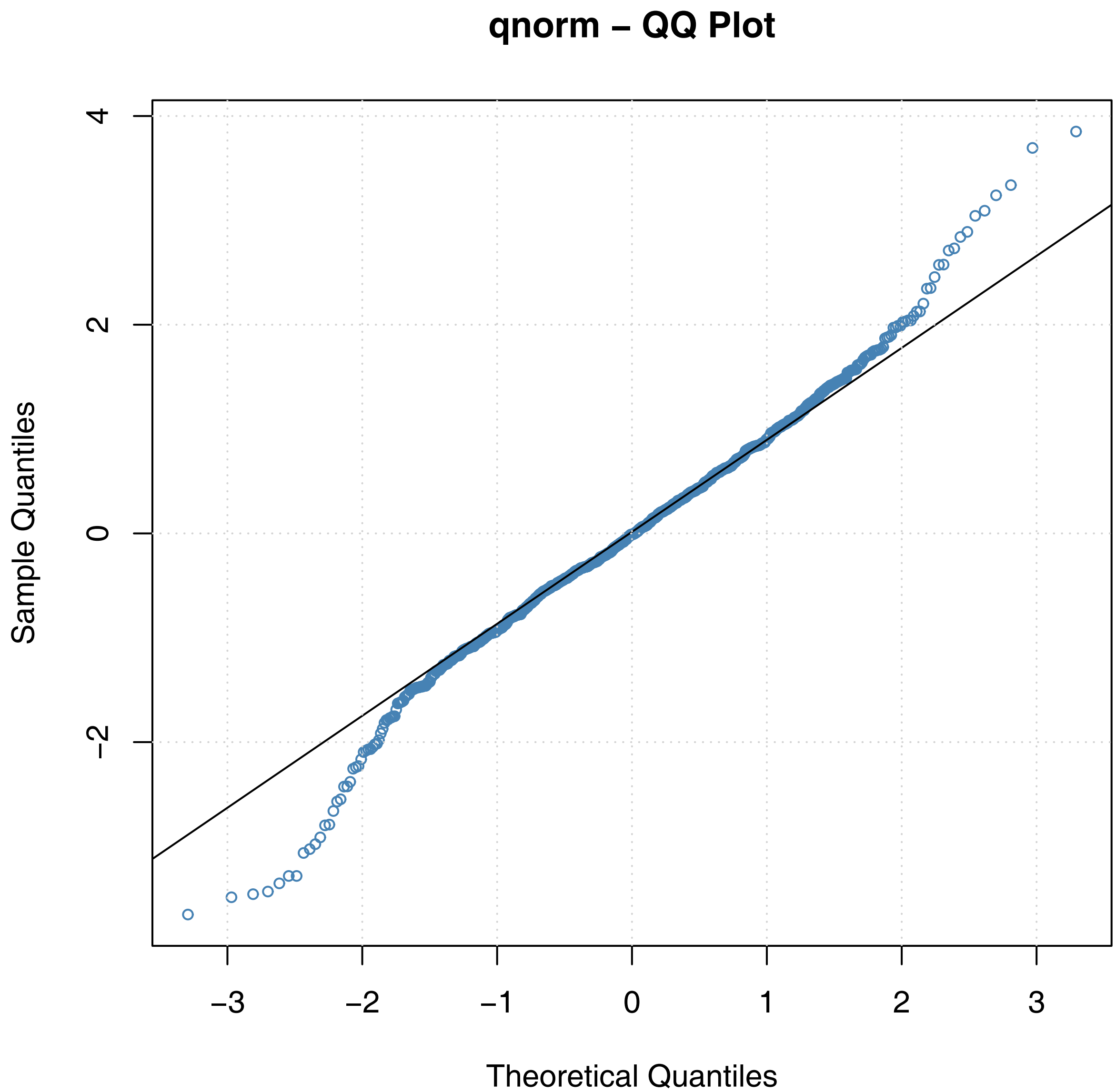

Figure 1: Normal probability plot for TGARCH(1,1) model fitted to monthly percentage log returns of IBM stock from 1926 to 2009

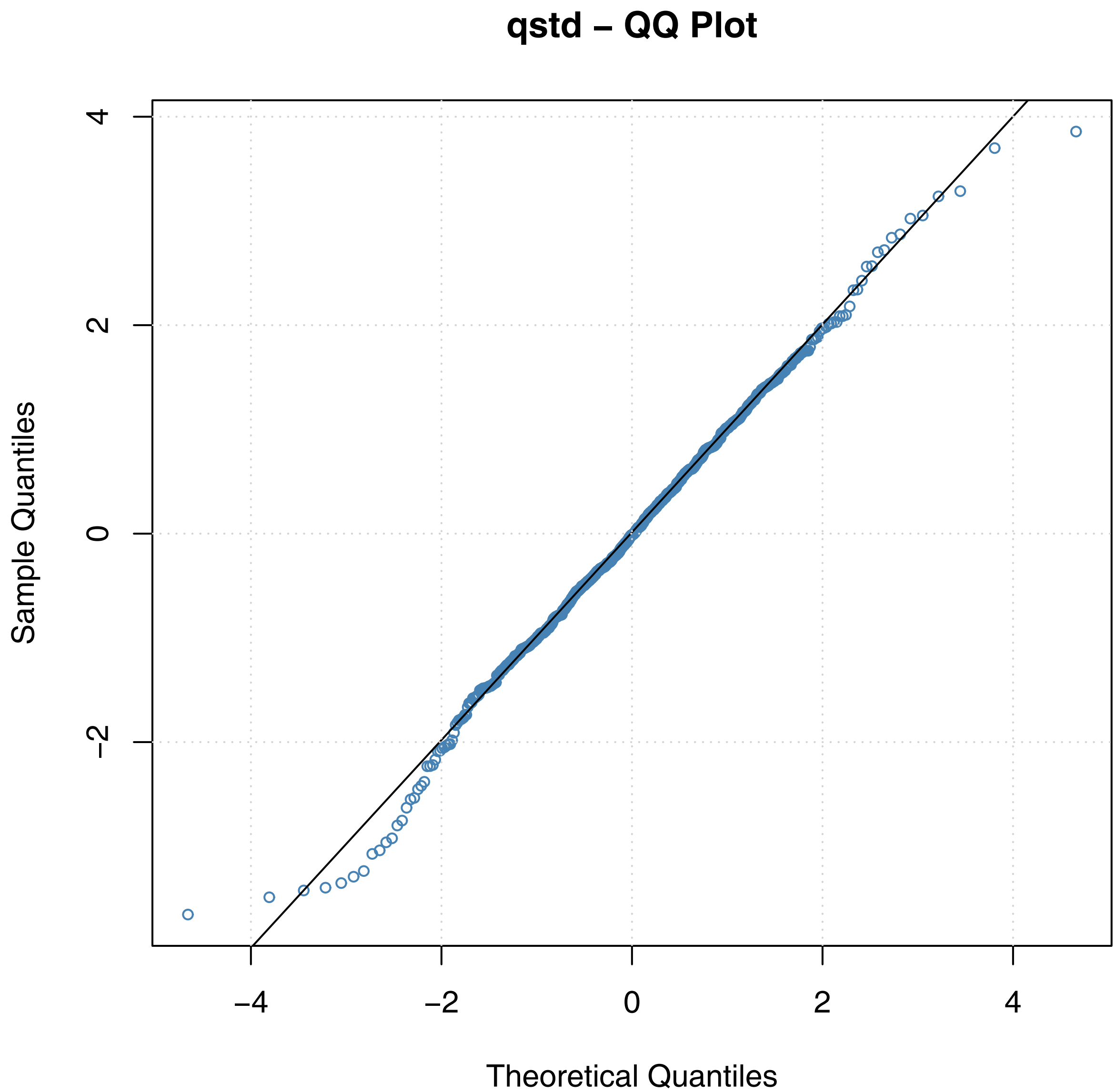

Figure 2: QQ plot for TGARCH(1,1) model fitted to monthly percentage log returns of IBM stock from 1926 to 2009.

A (simple) SV model is

$$a_t = \sigma_t \epsilon_t, \quad (1 - \alpha_1 B)\ln(\sigma_t^2) = \alpha_0 + v_t$$

where ϵ_t's are iid $N(0,1)$, v_t's are iid $N(0, \sigma_v^2)$, $\{\epsilon_t\}$ and $\{v_t\}$ are independent.

Estimation of SV model. May use the R package `stochvol` to estimate SV models.

Long-memory SV model

A simple LMSV is

$$a_t = \sigma_t \epsilon_t, \qquad \sigma_t = \sigma \exp(u_t/2), \qquad (1 - B)^d u_t = \eta_t$$

where $\sigma > 0$, ϵ_t's are iid $N(0,1)$, η_t's are iid $N(0, \sigma_\eta^2)$ and independent of ϵ_t, and $0 < d < 0.5$.

The model says

$$\begin{aligned} \ln(a_t^2) &= \ln(\sigma^2) + u_t + \ln(\epsilon_t^2) \\ &= [\ln(\sigma^2) + E(\ln \epsilon_t^2)] + u_t + [\ln(\epsilon_t^2) - E(\ln \epsilon_t^2)] \\ &\equiv \mu + u_t + e_t. \end{aligned}$$

Thus, the $\ln(a_t^2)$ series is a Gaussian long-memory signal plus a non-Gaussian white noise; see Breidt, Crato and de Lima (1998).

Application

See Examples 3.4 & 3.5 of the textbook