

**Financial Time Series**  
**Lecture 4: Univariate Volatility Models**  
**Conditional Heteroscedastic Models**

What is the volatility of an asset?

Answer: Conditional standard deviation of the asset return (price)

Why is volatility important?

Has many important applications:

- Option (derivative) pricing, e.g., Black-Scholes formula
- Risk management, e.g. value at risk (VaR)
- Asset allocation, e.g., minimum-variance portfolio; see pages 184-185 of Campbell, Lo and MacKinlay (1997).
- Interval forecasts

**A key characteristic:** Not directly observable!!

**How to calculate volatility?**

There are several versions of **sample volatility**, the conditional standard deviation being the most commonly used.

1. Use high-frequency data: French, Schwert & Stambaugh (1987); see Section 3.15.
  - Realized volatility of daily log returns: use intraday high-frequency log returns.
  - Use daily high, low, and closing (log) prices, e.g. range = daily high - daily low.
2. Implied volatility of options data, e.g, VIX of CBOE. Figure 1.

3. Econometric modeling: use daily or monthly returns

We focus on the econometric modeling first. Use of high frequency data to compute realized volatility will be discussed later.

**Note:** In most applications, volatility is annualized. This can easily be done by taking care of the data frequency. For instance, if we use daily returns in econometric modeling, then the annualized volatility (in the U.S.) is

$$\sigma_t^* = \sqrt{252}\sigma_t,$$

where  $\sigma_t$  is the estimated volatility derived from an employed model. If we use monthly returns, then the annualized volatility is

$$\sigma_t^* = \sqrt{12}\sigma_t,$$

where  $\sigma_t$  is the estimated volatility derived from the employed model for the monthly returns. Our discussion, however, continues to use  $\sigma_t$  for simplicity.

**Basic idea** of econometric modeling:

Shocks of asset returns are **NOT** serially correlated, but dependent, implying that the serial dependence in asset returns is nonlinear.

As shown by the ACF of returns and absolute returns of some assets we discussed so far.

**Basic structure**

$$r_t = \mu_t + a_t, \quad \mu_t = \phi_0 + \sum_{i=1}^p \phi_i r_{t-i} - \sum_{i=1}^q \theta_i a_{t-i},$$

Volatility models are concerned with time-evolution of

$$\sigma_t^2 = \text{Var}(r_t | F_{t-1}) = \text{Var}(a_t | F_{t-1}),$$

the conditional variance of the return  $r_t$ .

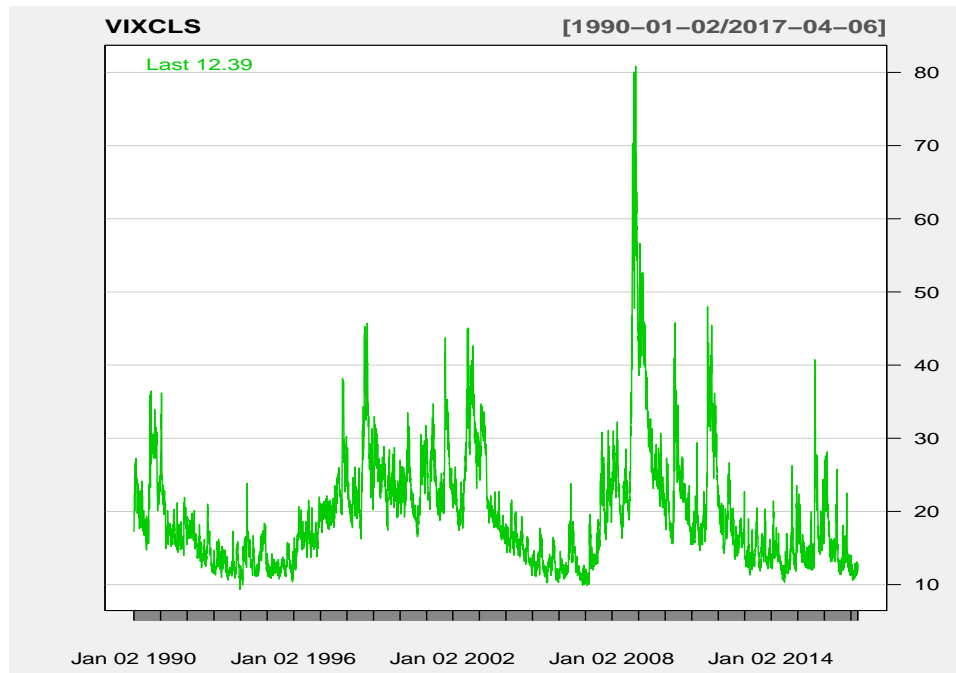


Figure 1: Time plot of the daily VIX index from January 2, 1990 to April 6, 2017.

Consider the daily closing index of the S&P500 index from January 03, 2007 to April 06, 2017. The log returns follow approximately an MA(2) model

$$r_t = 0.0002 + a_t - 0.109a_{t-1} - 0.053a_{t-2}, \quad \sigma^2 = 0.00017.$$

The residuals show no strong serial correlations. [plot not shown.]

### ***R Demonstration***

```
> require(quantmod)
> getSymbols("^GSPC",from="2007-01-03",to="2017-04-06")
[1] "GSPC"
> dim(GSPC)
[1] 2584    6
> head(GSPC)
      GSPC.Open GSPC.High GSPC.Low GSPC.Close GSPC.Volume GSPC.Adjusted
2007-01-03  1418.03  1429.42  1407.86   1416.60  3429160000    1416.60
..
2007-01-10  1408.70  1415.99  1405.32   1414.85  2764660000    1414.85
> spc <- log(as.numeric(GSPC[,6]))
> rtn <- diff(spc)
> acf(rtn)
```

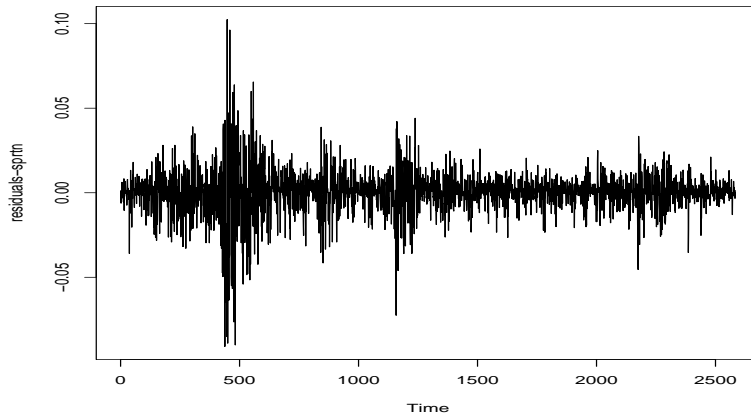


Figure 2: Time plot of residuals of an MA(2) model fitted to daily log returns of the S&P 500 index from January 3, 2007 to April 06, 2017.

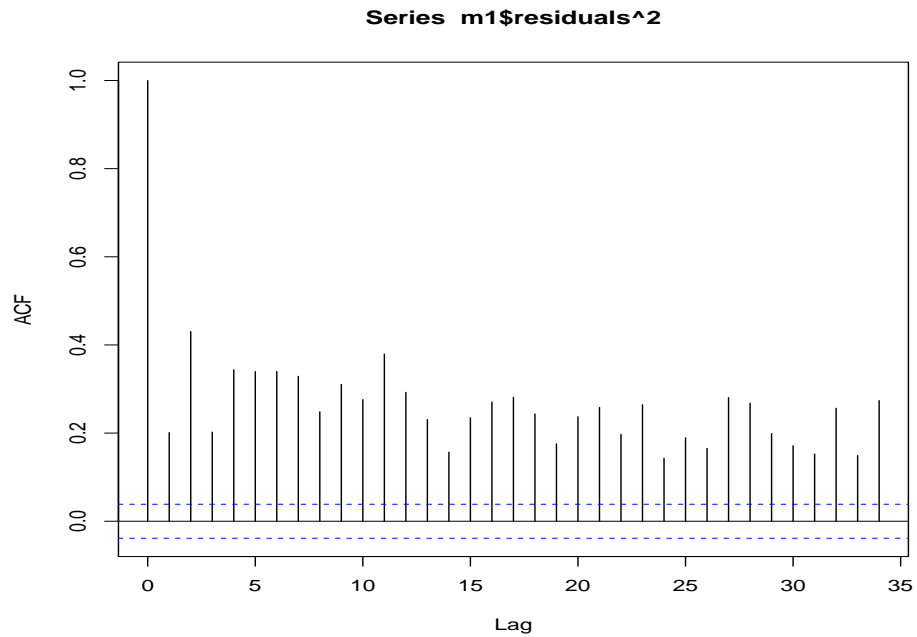


Figure 3: Sample ACF of the squared residuals of an MA(2) model fitted to daily log returns of the S&P 500 index from January 3, 2007 to April 06, 2017.

```

> m1 <- arima(rtn,order=c(0,0,2))
> m1
Call:  arima(x = rtn, order = c(0, 0, 2))

Coefficients:
          ma1          ma2  intercept
      -0.1088  -0.0530         2e-04
s.e.    0.0198   0.0205         2e-04

sigma^2 estimated as 0.0001676:  log likelihood = 7562.82,  aic = -15117.65
> acf(m1$residuals)
> acf(m1$residuals^2)

```

The residuals are shown in Figure 2.

Is volatility constant over time?

NO! Figure 2 shows a special feature, which is referred to as the **volatility clustering**.

How to model the evolving volatility? See the ACF of the squared residuals in Figure 3.

## Two general categories

- “Fixed function” and
- Stochastic function

of the available information.

## Univariate volatility models discussed:

1. Autoregressive conditional heteroscedastic (ARCH) model of Engle (1982),
2. Generalized ARCH (GARCH) model of Bollerslev (1986),
3. GARCH-M models,
4. IGARCH models (used by RiskMetrics),
5. Exponential GARCH (EGARCH) model of Nelson (1991),

6. Threshold GARCH model of Zakoian (1994) or GJR model of Glosten, Jagannathan, and Runkle (1993),
7. Asymmetric power ARCH (APARCH) models of Ding, Granger and Engle (1994), [TGARCH and GJR models are special cases of APARCH models.]
8. Stochastic volatility (SV) models of Melino and Turnbull (1990), Harvey, Ruiz and Shephard (1994), and Jacquier, Polson and Rossi (1994).

## ARCH model

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2,$$

where  $\{\epsilon_t\}$  is a sequence of iid r.v. with mean 0 and variance 1,  $\alpha_0 > 0$  and  $\alpha_i \geq 0$  for  $i > 0$ .

Distribution of  $\epsilon_t$ : Standard normal, standardized Student-t, generalized error dist (ged), or their skewed counterparts.

## Properties of ARCH models

Consider an ARCH(1) model

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2,$$

where  $\alpha_0 > 0$  and  $\alpha_1 \geq 0$ .

1.  $E(a_t) = 0$
2.  $\text{Var}(a_t) = \alpha_0 / (1 - \alpha_1)$  if  $0 < \alpha_1 < 1$
3. Under normality,

$$m_4 = \frac{3\alpha_0^2(1 + \alpha_1)}{(1 - \alpha_1)(1 - 3\alpha_1^2)},$$

provided  $0 < \alpha_1^2 < 1/3$ .

The 3rd property implies heavy tails.

## Advantages

- Simplicity
- Generates volatility clustering
- Heavy tails (high kurtosis)

## Weaknesses

- Symmetric between positive & negative prior returns
- Restrictive
- Provides no explanation
- Not sufficiently adaptive in prediction

## Building an ARCH Model

1. Modeling the mean effect and testing for ARCH effects

$H_0$ : no ARCH effects versus  $H_a$  : ARCH effects

Use Q-statistics of squared residuals; McLeod and Li (1983) & Engle (1982)

2. Order determination

Use PACF of the squared residuals. (In practice, simply try some reasonable order).

3. Estimation: Conditional MLE

4. Model checking: Q-stat of standardized residuals and squared standardized residuals. Skewness & Kurtosis of standardized residuals.

R provides many plots for model checking and for presenting the results.

5. Software: We use R with the package `fGarch`. (Other software available).

**Estimation:** Conditional MLE or Quasi MLE

**Special Note:** In this course, we estimate volatility models using the R package `fGarch` with `garchFit` command. The program is easy to use and allows for several types of innovational distributions: The default is Gaussian (**norm**), standardized Student-*t* distribution (**std**), generalized error distribution (**ged**), skew normal distribution (**snorm**), skew Student-*t* (**sstd**), skew generalized error distribution (**sged**), and standardized inverse normal distribution (**snig**). Except for the inverse normal distribution, other distribution functions are discussed in the textbook. Readers should check the book for details about the density functions and their parameters.

**Example:** Monthly log returns of Intel stock

**R demonstration:** The `fGarch` package. Output edited.

```
> library(fGarch)
> da=read.table("m-intc7303.txt",header=T)
> head(da)
      date      rtn
1 19730131 0.01005
.....
6 19730629 0.13333
> intc=log(da$rtn+1) <== log returns
> acf(intc)
> acf(intc^2)
> pacf(intc^2)
> Box.test(intc^2,lag=10,type='Ljung')
      Box-Ljung test

data:  intc^2
X-squared = 59.7216, df = 10, p-value = 4.091e-09
```



```
> m1=garchFit(~garch(3,0),data=intc,trace=F) <== trace=F reduces the amount of output.
> summary(m1)
```

Title: GARCH Modelling

Call: garchFit(formula = ~garch(3, 0), data = intc, trace = F)

Mean and Variance Equation:

```
data ~ garch(3, 0)
[data = intc]
```

Conditional Distribution: norm

Coefficient(s):

	mu	omega	alpha1	alpha2	alpha3
	0.016572	0.012043	0.208649	0.071837	0.049045

Std. Errors:

based on Hessian

Error Analysis:

	Estimate	Std. Error	t value	Pr(> t )
mu	0.016572	0.006423	2.580	0.00988 **
omega	0.012043	0.001579	7.627	2.4e-14 ***
alpha1	0.208649	0.129177	1.615	0.10626
alpha2	0.071837	0.048551	1.480	0.13897
alpha3	0.049045	0.048847	1.004	0.31536

---

Standardised Residuals Tests:

			Statistic	p-Value
Jarque-Bera Test	R	Chi <sup>2</sup>	169.7731	0
Shapiro-Wilk Test	R	W	0.9606957	1.970413e-08
Ljung-Box Test	R	Q(10)	10.97025	0.3598405
Ljung-Box Test	R	Q(15)	19.59024	0.1882211
Ljung-Box Test	R	Q(20)	20.82192	0.40768
Ljung-Box Test	R <sup>2</sup>	Q(10)	5.376602	0.864644
Ljung-Box Test	R <sup>2</sup>	Q(15)	22.73460	0.08993976
Ljung-Box Test	R <sup>2</sup>	Q(20)	23.70577	0.255481
LM Arch Test	R	TR <sup>2</sup>	20.48506	0.05844884

Information Criterion Statistics:

AIC	BIC	SIC	HQIC
-1.228111	-1.175437	-1.228466	-1.207193

```
> m1=garchFit(~garch(1,0),data=intc,trace=F)
```

```
> summary(m1)
```

Title: GARCH Modelling

Call: garchFit(formula = ~garch(1, 0), data = intc, trace = F)

Mean and Variance Equation:

data ~ garch(1, 0)  
[data = intc]

Conditional Distribution: norm

Coefficient(s):

	mu	omega	alpha1
	0.016570	0.012490	0.363447

Std. Errors:

based on Hessian

Error Analysis:

	Estimate	Std. Error	t value	Pr(> t )
mu	0.016570	0.006161	2.689	0.00716 **
omega	0.012490	0.001549	8.061	6.66e-16 ***
alpha1	0.363447	0.131598	2.762	0.00575 **

---

Log Likelihood:

230.2423      normalized: 0.6189309

Standardised Residuals Tests:

			Statistic	p-Value	
Jarque-Bera Test	R	Chi <sup>2</sup>	122.4040	0	
Shapiro-Wilk Test	R	W	0.9647629	8.274158e-08	
Ljung-Box Test	R	Q(10)	13.72604	0.1858587	<=== Meaning?
Ljung-Box Test	R	Q(15)	22.31714	0.09975386	<==== implication?
Ljung-Box Test	R	Q(20)	23.88257	0.2475594	
Ljung-Box Test	R <sup>2</sup>	Q(10)	12.50025	0.2529700	
Ljung-Box Test	R <sup>2</sup>	Q(15)	30.11276	0.01152131	
Ljung-Box Test	R <sup>2</sup>	Q(20)	31.46404	0.04935483	
LM Arch Test	R	TR <sup>2</sup>	22.036	0.0371183	

Information Criterion Statistics:

AIC	BIC	SIC	HQIC
-1.221733	-1.190129	-1.221861	-1.209182

> plot(m1)

Make a plot selection (or 0 to exit):

- 1: Time Series
- 2: Conditional SD
- 3: Series with 2 Conditional SD Superimposed

- 4: ACF of Observations
- 5: ACF of Squared Observations
- 6: Cross Correlation
- 7: Residuals
- 8: Conditional SDs
- 9: Standardized Residuals
- 10: ACF of Standardized Residuals
- 11: ACF of Squared Standardized Residuals
- 12: Cross Correlation between  $r^2$  and  $r$
- 13: QQ-Plot of Standardized Residuals

Selection: 13

Make a plot selection (or 0 to exit):

- 1: Time Series
- 2: Conditional SD
- 3: Series with 2 Conditional SD Superimposed
- 4: ACF of Observations
- 5: ACF of Squared Observations
- 6: Cross Correlation
- 7: Residuals
- 8: Conditional SDs
- 9: Standardized Residuals
- 10: ACF of Standardized Residuals
- 11: ACF of Squared Standardized Residuals
- 12: Cross Correlation between  $r^2$  and  $r$
- 13: QQ-Plot of Standardized Residuals

Selection: 0

The fitted ARCH(1) model is

$$r_t = 0.0176 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1)$$

$$\sigma_t^2 = 0.0125 + 0.363\sigma_{t-1}^2.$$

Model checking statistics indicate that there are some higher order dependence in the volatility, e.g., see Q(15) for the squared standardized residuals. It turns out that a GARCH(1,1) model fares better for the data.

Next, consider Student-t innovations.

## **R demonstration**

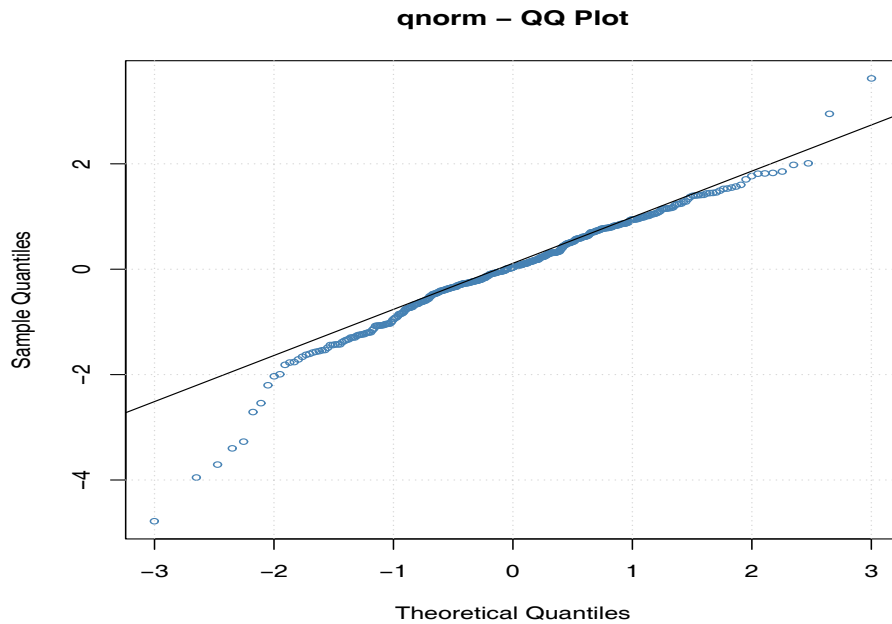


Figure 4: QQ-plot for standardized residuals of an ARCH(1) model with Gaussian innovations for monthly log returns of INTC stock: 1973 to 2003.

```

> m2=garchFit(~garch(1,0),data=intc,cond.dist="std",trace=F)
> summary(m2)
Title: GARCH Modelling

Call: garchFit(formula = ~garch(1, 0), data = intc, cond.dist = "std",
               trace = F)

Mean and Variance Equation:
data ~ garch(1, 0)
[data = intc]

Conditional Distribution:  std <==== Standardized Student-t.

Coefficient(s):
      mu      omega    alpha1    shape
0.021571 0.013424 0.259867 5.985979

Error Analysis:
      Estimate Std. Error t value Pr(>|t|)
mu      0.021571  0.006054   3.563 0.000366 ***
omega   0.013424  0.001968   6.820 9.09e-12 ***
alpha1  0.259867  0.119901   2.167 0.030209 *

```

```
shape 5.985979 1.660030 3.606 0.000311 *** <== Estimate of degrees of freedom
---
```

```
Log Likelihood:
```

```
242.9678 normalized: 0.6531391
```

```
Standardised Residuals Tests:
```

			Statistic	p-Value
Jarque-Bera Test	R	Chi <sup>2</sup>	130.8931	0
Shapiro-Wilk Test	R	W	0.9637529	5.744026e-08
Ljung-Box Test	R	Q(10)	14.31288	0.1591926
Ljung-Box Test	R	Q(15)	23.34043	0.07717449
Ljung-Box Test	R	Q(20)	24.87286	0.2063387
Ljung-Box Test	R <sup>2</sup>	Q(10)	15.35917	0.1195054
Ljung-Box Test	R <sup>2</sup>	Q(15)	33.96318	0.003446127
Ljung-Box Test	R <sup>2</sup>	Q(20)	35.46828	0.01774746
LM Arch Test	R	TR <sup>2</sup>	24.11517	0.01961957

```
Information Criterion Statistics:
```

AIC	BIC	SIC	HQIC
-1.284773	-1.242634	-1.285001	-1.268039

```
> plot(m2)
```

```
Make a plot selection (or 0 to exit):
```

- 1: Time Series
- 2: Conditional SD
- 3: Series with 2 Conditional SD Superimposed
- 4: ACF of Observations
- 5: ACF of Squared Observations
- 6: Cross Correlation
- 7: Residuals
- 8: Conditional SDs
- 9: Standardized Residuals
- 10: ACF of Standardized Residuals
- 11: ACF of Squared Standardized Residuals
- 12: Cross Correlation between r<sup>2</sup> and r
- 13: QQ-Plot of Standardized Residuals

```
Selection: 13 <== The plot shows that the model needs further improvements.
```

```
> predict(m2,5) <===== Prediction
```

	meanForecast	meanError	standardDeviation
1	0.02157100	0.1207911	0.1207911
2	0.02157100	0.1312069	0.1312069
3	0.02157100	0.1337810	0.1337810
4	0.02157100	0.1344418	0.1344418

5 0.02157100 0.1346130 0.1346130

The fitted model with Student- $t$  innovations is

$$\begin{aligned}r_t &= 0.0216 + a_t, & a_t &= \sigma_t \epsilon_t, & \epsilon &\sim t_{5.99} \\ \sigma_t^2 &= 0.0134 + 0.260a_{t-1}^2.\end{aligned}$$

We use  $t_{5.99}$  to denote the standardized Student- $t$  distribution with 5.99 d.f.

Comparison with normal innovations:

- Using a heavy-tailed dist for  $\epsilon_t$  reduces the ARCH effect.
- The difference between the models is small for this particular instance.

You may try other distributions for  $\epsilon_t$ .

## GARCH Model

$$\begin{aligned}a_t &= \sigma_t \epsilon_t, \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^m \alpha_i a_{t-i}^2 + \sum_{j=1}^s \beta_j \sigma_{t-j}^2\end{aligned}$$

where  $\{\epsilon_t\}$  is defined as before,  $\alpha_0 > 0$ ,  $\alpha_i \geq 0$ ,  $\beta_j \geq 0$ , and  $\sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1$ .

Re-parameterization:

Let  $\eta_t = a_t^2 - \sigma_t^2$ .  $\{\eta_t\}$  un-correlated series.

The GARCH model becomes

$$a_t^2 = \alpha_0 + \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) a_{t-i}^2 + \eta_t - \sum_{j=1}^s \beta_j \eta_{t-j}.$$

This is an ARMA form for the squared series  $a_t^2$ .

Use it to understand properties of GARCH models, e.g. moment equations, forecasting, etc.

Focus on a GARCH(1,1) model

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

- Weak stationarity:  $0 \leq \alpha_1, \beta_1 \leq 1, (\alpha_1 + \beta_1) < 1$ .
- Volatility clusters
- Heavy tails: if  $1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2 > 0$ , then

$$\frac{E(a_t^4)}{[E(a_t^2)]^2} = \frac{3[1 - (\alpha_1 + \beta_1)^2]}{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2} > 3.$$

- For 1-step ahead forecast,

$$\sigma_h^2(1) = \alpha_0 + \alpha_1 a_h^2 + \beta_1 \sigma_h^2.$$

For multi-step ahead forecasts, use  $a_t^2 = \sigma_t^2 \epsilon_t^2$  and rewrite the model as

$$\sigma_{t+1}^2 = \alpha_0 + (\alpha_1 + \beta_1) \sigma_t^2 + \alpha_1 \sigma_t^2 (\epsilon_t^2 - 1).$$

2-step ahead volatility forecast

$$\sigma_h^2(2) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2(1).$$

In general, we have

$$\sigma_h^2(\ell) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2(\ell - 1), \quad \ell > 1.$$

This result is exactly the same as that of an ARMA(1,1) model with AR polynomial  $1 - (\alpha_1 + \beta_1)B$ .

**Example:** Monthly excess returns of S&P 500 index starting from 1926 for 792 observations.

The fitted of a Gaussian AR(3) model

$$\tilde{r}_t = r_t - 0.0062$$

$$\tilde{r}_t = .089\tilde{r}_{t-1} - .024\tilde{r}_{t-2} - .123\tilde{r}_{t-3} + .007 + a_t,$$

$$\hat{\sigma}_a^2 = 0.00333.$$

For the GARCH effects, use a GARCH(1,1) model, we have

A joint estimation:

$$r_t = 0.032r_{t-1} - 0.030r_{t-2} - 0.011r_{t-3} + 0.0077 + a_t$$

$$\sigma_t^2 = 7.98 \times 10^{-5} + .853\sigma_{t-1}^2 + 0.124a_{t-1}^2.$$

Implied unconditional variance of  $a_t$  is

$$\frac{0.0000798}{1 - 0.853 - 0.1243} = 0.00352$$

close to the expected value. All AR coefficients are statistically insignificant.

A simplified model:

$$r_t = 0.00745 + a_t, \sigma_t^2 = 8.06 \times 10^{-5} + .854\sigma_{t-1}^2 + .122a_{t-1}^2.$$

Model checking:

For  $\tilde{a}_t$ :  $Q(10) = 11.22(0.34)$  and  $Q(20) = 24.30(0.23)$ .

For  $\tilde{a}_t^2$ :  $Q(10) = 9.92(0.45)$  and  $Q(20) = 16.75(0.67)$ .

Forecast: 1-step ahead forecast:

$$\sigma_h^2(1) = 0.00008 + 0.854\sigma_h^2 + 0.122a_h^2$$

Horizon	1	2	3	4	5	$\infty$
Return	.0074	.0074	.0074	.0074	.0074	.0074
Volatility	.054	.054	.054	.054	.054	.059

***R demonstration:***



```

> sp5=scan("sp500.txt")
Read 792 items
> pacf(sp5)
> m1=arima(sp5,order=c(3,0,0))
> m1
Call: arima(x = sp5, order = c(3, 0, 0))

Coefficients:
      ar1      ar2      ar3  intercept
      0.0890 -0.0238 -0.1229      0.0062
s.e.  0.0353   0.0355   0.0353      0.0019

sigma^2 estimated as 0.00333:  log likelihood = 1135.25, aic=-2260.5
> m2=garchFit(~arma(3,0)+garch(1,1),data=sp5,trace=F)
> summary(m2)
Title: GARCH Modelling
Call:
garchFit(formula = ~arma(3,0)+garch(1, 1), data = sp5, trace = F)

Mean and Variance Equation:
data ~ arma(3, 0) + garch(1, 1)
[data = sp5]

Conditional Distribution: norm

Error Analysis:
      Estimate Std. Error t value Pr(>|t|)
mu      7.708e-03  1.607e-03   4.798 1.61e-06 ***
ar1     3.197e-02  3.837e-02   0.833 0.40473
ar2    -3.026e-02  3.841e-02  -0.788 0.43076
ar3    -1.065e-02  3.756e-02  -0.284 0.77677
omega   7.975e-05  2.810e-05   2.838 0.00454 **
alpha1  1.242e-01  2.247e-02   5.529 3.22e-08 ***
beta1   8.530e-01  2.183e-02  39.075 < 2e-16 ***
---
Log Likelihood:
1272.179    normalized: 1.606287

Standardised Residuals Tests:
      Statistic p-Value
Jarque-Bera Test  R  Chi^2  73.04842  1.110223e-16
Shapiro-Wilk Test R  W      0.985797  5.961994e-07
Ljung-Box Test   R  Q(10) 11.56744  0.315048
Ljung-Box Test   R  Q(15) 17.78747  0.2740039
Ljung-Box Test   R  Q(20) 24.11916  0.2372256
Ljung-Box Test   R^2 Q(10) 10.31614  0.4132089

```

```

Ljung-Box Test      R^2  Q(15)  14.22819  0.5082978
Ljung-Box Test      R^2  Q(20)  16.79404  0.6663038
LM Arch Test        R     TR^2   13.34305  0.3446075

```

Information Criterion Statistics:

```

      AIC      BIC      SIC      HQIC
-3.194897 -3.153581 -3.195051 -3.179018

```

```
> m2=garchFit(~garch(1,1),data=sp5,trace=F)
```

```
> summary(m2)
```

Title: GARCH Modelling

Call: garchFit(formula = ~garch(1, 1), data = sp5, trace = F)

Mean and Variance Equation:

```

data ~ garch(1, 1)
[data = sp5]

```

Conditional Distribution: norm

Error Analysis:

```

      Estimate  Std. Error  t value  Pr(>|t|)
mu      7.450e-03  1.538e-03   4.845  1.27e-06 ***
omega   8.061e-05  2.833e-05   2.845  0.00444 **
alpha1  1.220e-01  2.202e-02   5.540  3.02e-08 ***
beta1   8.544e-01  2.175e-02  39.276  < 2e-16 ***
---

```

Log Likelihood:

```

1269.455    normalized:  1.602848

```

Standardised Residuals Tests:

```

      Statistic  p-Value
Jarque-Bera Test  R     Chi^2  80.32111  0
Shapiro-Wilk Test  R     W      0.9850517  3.141228e-07
Ljung-Box Test     R     Q(10)  11.22050  0.340599
Ljung-Box Test     R     Q(15)  17.99703  0.262822
Ljung-Box Test     R     Q(20)  24.29896  0.2295768
Ljung-Box Test     R^2   Q(10)  9.920157  0.4475259
Ljung-Box Test     R^2   Q(15)  14.21124  0.509572
Ljung-Box Test     R^2   Q(20)  16.75081  0.6690903
LM Arch Test       R     TR^2   13.04872  0.3655092

```

Information Criterion Statistics:

```

      AIC      BIC      SIC      HQIC
-3.195594 -3.171985 -3.195645 -3.186520

```

```
> plot(m2)
```

Make a plot selection (or 0 to exit):

- 1: Time Series
- 2: Conditional SD
- 3: Series with 2 Conditional SD Superimposed
- 4: ACF of Observations
- 5: ACF of Squared Observations
- 6: Cross Correlation
- 7: Residuals
- 8: Conditional SDs
- 9: Standardized Residuals
- 10: ACF of Standardized Residuals
- 11: ACF of Squared Standardized Residuals
- 12: Cross Correlation between  $r^2$  and  $r$
- 13: QQ-Plot of Standardized Residuals

Selection: 3

```
> predict(m2,6)
  meanForecast meanError standardDeviation
1 0.007449721 0.05377242      0.05377242
2 0.007449721 0.05388567      0.05388567
3 0.007449721 0.05399601      0.05399601
4 0.007449721 0.05410353      0.05410353
5 0.007449721 0.05420829      0.05420829
6 0.007449721 0.05431038      0.05431038
```

Turn to Student- $t$  innovation. (R output omitted.)

Estimation of degrees of freedom:

$$r_t = 0.0085 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t_7$$
$$\sigma_t^2 = .000125 + .113a_{t-1}^2 + .842\sigma_{t-1}^2,$$

where the estimated degrees of freedom is 7.00.

## Forecasting evaluation

Not easy to do; see Andersen and Bollerslev (1998).

## IGARCH model

An IGARCH(1,1) model:

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1 - \beta_1) a_{t-1}^2.$$

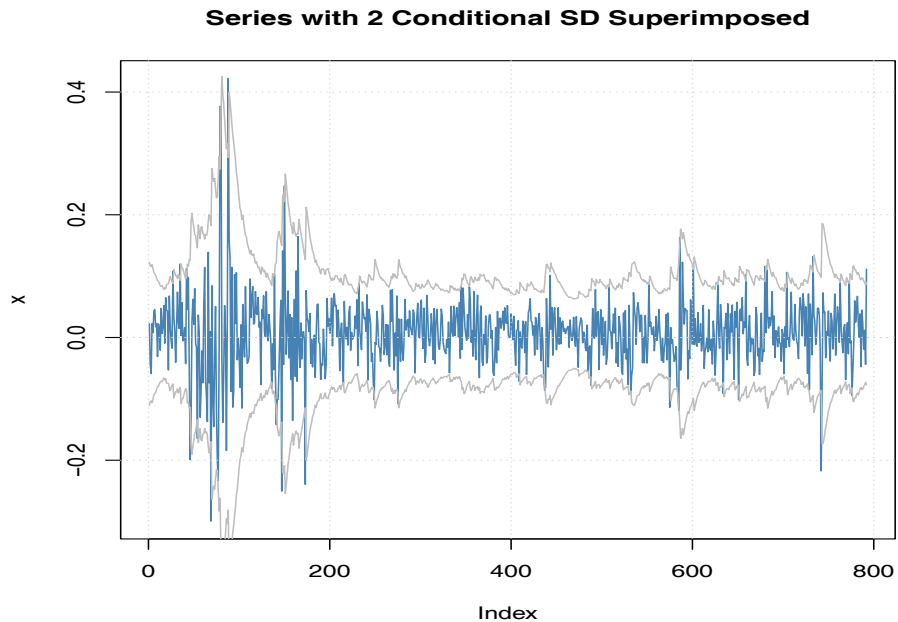


Figure 5: Monthly S&P 500 excess returns and fitted volatility

For the monthly excess returns of the S&P 500 index, we have

$$r_t = .007 + a_t, \sigma_t^2 = .0001 + .806\sigma_{t-1}^2 + .194a_{t-1}^2$$

For an IGARCH(1,1) model,

$$\sigma_h^2(\ell) = \sigma_h^2(1) + (\ell - 1)\alpha_0, \quad \ell \geq 1,$$

where  $h$  is the forecast origin.

Effect of  $\sigma_h^2(1)$  on future volatilities is persistent, and the volatility forecasts form a straight line with slope  $\alpha_0$ . See Nelson (1990) for more info.

Special case:  $\alpha_0 = 0$ . Volatility forecasts become a constant.

This property is used in RiskMetrics to VaR calculation.

**Example:** An IGARCH(1,1) model for the monthly excess returns of S&P500 index from 1926 to 1991 is given below via R.

$$r_t = 0.0069 + a_t, \quad a_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = 0.099a_{t-1}^2 + .901\sigma_{t-1}^2.$$

## R demonstration: Using R script Igarch.R.

```
> source("Igarch.R")
> sp5=scan(file="sp500.txt")
> Igarch(sp5,include.mean=T)
Estimates:  0.006874402 0.9007153
Maximized log-likelihood: -1258.219
```

Coefficient(s):

	Estimate	Std. Error	t value	Pr(> t )
mu	0.0068744	0.0015402	4.46332	8.07e-06 ***
beta	0.9007153	0.0158018	57.00082	< 2e-16 ***

**Another R package:** rugarch can be used to fit volatility models too.

```
> sp5=scan("sp500.txt")
> require(rugarch)
> spec1=ugarchspec(variance.model=list(model="iGARCH",garchOrder=c(1,1)),
                  mean.model=list(armaOrder=c(0,0)))
> mm=ugarchfit(data=sp5,spec=spec1)
> mm
```

```
*-----*
*          GARCH Model Fit          *
*-----*
```

Conditional Variance Dynamics

```
-----
GARCH Model      : iGARCH(1,1)
Mean Model       : ARFIMA(0,0,0)
Distribution      : norm
```

Optimal Parameters

	Estimate	Std. Error	t value	Pr(> t )
mu	0.007417	0.001525	4.8621	0.000001
omega	0.000051	0.000018	2.9238	0.003458
alpha1	0.142951	0.021443	6.6667	0.000000
beta1	0.857049	NA	NA	NA

Robust Standard Errors:

	Estimate	Std. Error	t value	Pr(> t )
mu	0.007417	0.001587	4.6726	0.000003
omega	0.000051	0.000019	2.6913	0.007118
alpha1	0.142951	0.024978	5.7230	0.000000
beta1	0.857049	NA	NA	NA

LogLikelihood : 1268.238

Information Criteria

-----  
Akaike           -3.1950  
Bayes            -3.1773  
Shibata          -3.1951  
Hannan-Quinn   -3.1882

Weighted Ljung-Box Test on Standardized Residuals

-----  
                          statistic p-value  
Lag[1]                    0.5265  0.4681  
Lag[2\*(p+q)+(p+q)-1] [2]  0.5304  0.6795  
Lag[4\*(p+q)+(p+q)-1] [5]  2.5233  0.5009  
d.o.f=0  
H0 : No serial correlation

Weighted Ljung-Box Test on Standardized Squared Residuals

-----  
                          statistic p-value  
Lag[1]                    1.166  0.2803  
Lag[2\*(p+q)+(p+q)-1] [5]  2.672  0.4702  
Lag[4\*(p+q)+(p+q)-1] [9]  4.506  0.5054  
d.o.f=2  
.....