STAT 5380 Assignment 7:
Testing theory, Neyman Pearson, UMP, UMPU, UMA, LR/Score/Wald tests

1. Variation on TSH Problem 3.2 as follows. For $X_1, \dots, X_n$ a random sample from a uniform on $(0, \theta)$:
   (i) Find a UMP level α test of $H : \theta \le \theta_0$ vs. $K : \theta > \theta_0$, and (compute and) sketch its power function.
   (ii) Find a UMP level α test of $H : \theta \ge \theta_0$ vs. $K : \theta < \theta_0$, and (compute and) sketch its power function.
2. TSH Problem 3.6.
3. TSH Problem 3.26 (3rd ed)/3.27 (4th ed).
4. TSH Problem 3.29 (3rd ed)/3.30 (4th ed).
   (i) As stated. (Note that you should construct a UMP test here.)
   (ii) As stated.
   (iii) Construct UMA lower and upper level $\alpha = 0.05$ confidence bounds for θ.
   (iv) Use your UMA bounds to construct a level $1 - \alpha = 0.90$ confidence interval for θ.
5. TSH Problem 4.4.
6. TSH Problem 4.25.
7. Revisit TSH Problem 4.25, but this time construct the corresponding statistics for the LR, Wald, and Score tests, that would be used in the UMPU test in (i). Take the calculations as far as you can analytically, and indicate what would need to be computed from there. Does it look as if any of these tests coincides with the UMPU? (You can use Observed rather than Expected Information matrices if the calculations get too heavy.)

(Over for last question.)

8. (Long Gamma Problem.) Let $X_1, \ldots, X_n$ be iid from the distribution of X which is a Gamma(θ, λ), where $\theta > 0$ is the shape parameter, and $\lambda > 0$ is the rate parameter, so that the pdf of X is:

$$f(x;\theta,\lambda) = \frac{\lambda^\theta}{\Gamma(\theta)} x^{\theta-1} e^{-\lambda x} I_{(0,\infty)}(x).$$

Note that $\mu = \mathbb{E}(X) = \theta/\lambda$, $\mathbb{V}(X) = \theta/\lambda^2$, and $\mathbb{E}(X-\mu)^3 = 2\theta/\lambda^3$. Define the (sufficient) statistics:

$$U = \sum_{i=1}^n \log X_i, \qquad T = \sum_{i=1}^n X_i.$$

You will need to use the fact that derivatives of $\log \Gamma(y)$ are defined through the *polygamma* function as follows:

$$\frac{\partial \log \Gamma(y)}{\partial y} = \Psi(y), \quad \text{(digamma)}, \qquad \frac{\partial \Psi(y)}{\partial y} = \Psi(1,y), \quad \text{(1st order polygamma)}.$$

(a) Deduce that this is a 2-parameter (full-rank) exponential family in the parameters $\boldsymbol{\beta} = (\theta, \lambda)$, and hence show that the (expected) Information matrix (per observation) is:

$$I(\boldsymbol{\beta}) = \begin{pmatrix} \Psi(1,\theta) & 1/\lambda \\ 1/\lambda & \theta/\lambda^2 \end{pmatrix}.$$

(b) For the case $n = 2$: find the joint distribution of (U, T), the respective marginals U and T, and hence find the conditional distribution of $U|T$.

(c) For the case $n = 2$ and λ <u>known</u>: find the level α UMP test of $H : \theta \le \theta_0$ vs. $K : \theta > \theta_0$. (Explicitly determine the cutoff points of the test as functions of θ_0, α, and λ.)

(d) For the case $n = 2$ and λ <u>unknown</u>: find the level α UMPU test of $H : \theta \le \theta_0$ vs. $K : \theta > \theta_0$. (Explicitly determine the cutoff points of the test as functions of θ_0, α, and t.)

(e) For the general n case: construct the LR, Wald, and Score test statistics that would be used in testing $H : \theta = 1$ vs. $K : \theta \neq 1$.

(f) Either analytically (if possible) or by Monte Carlo simulation, compute the power of each of the tests in (e), when $n = 2$ and $\alpha = 0.05$. (Use the $\chi^2(1)$ approximation to the null, and assume the true value of $\lambda = 1$.)