STAT 5379 Assignment 2

(Covered in Homeworks 6–9)

Mathematical concepts and derivations

1. (B&D Problem 3.12) For the MA(1), $X_t = Z_t + \theta Z_{t-1}$, where $Z_t \sim \text{WN}(0, \sigma^2)$, the BLP of X_{n+1} based on $X_1, \ldots, X_n$ is
$$P_n X_{n+1} \equiv \hat{X}_{n+1} = \phi_{n,1} X_n + \ldots + \phi_{n,n} X_1,$$
where $\boldsymbol{\phi}_n = (\phi_{n,1}, \ldots, \phi_{n,n})$ satisfies: $R_n \boldsymbol{\phi}_n = \boldsymbol{\rho}$. By substituting the appropriate correlations into R_n and $\boldsymbol{\rho}$ and solving the resulting equations (starting with the last and working up), show that for $1 \leq j < n$,
$$\phi_{n,n-j} = (-\theta)^{-j}(1 + \theta^2 + \ldots + \theta^{2j})\phi_{n,n},$$
and hence that the PACF at lag n is:
$$\alpha(n) = \phi_{n,n} = -(-\theta)^n(1 + \theta^2 + \ldots + \theta^{2n}).$$

2. (S&S Problem 3.11.) Consider the MA(1) process $X_t = Z_t + \theta Z_{t-1}$, where $Z_t \sim \text{WN}(0, \sigma^2)$.
   (a) Derive the BLP one-step forecast based on the infinite past, $\tilde{P}_n X_{n+1}$, and determine the mean-square error of this forecast.
   (b) Let $\tilde{X}_{n+1}^n$ be the BLP one-step forecast based on the infinite past as in (a), but truncated at X_1 (i.e., we discard the portion of $\tilde{P}_n X_{n+1}$ containing X_t for $t \leq 0$.)

3. (S&S Problem 3.15.) Consider the AR(1) process $X_t = \phi X_{t-1} + Z_t$, where $Z_t \sim \text{WN}(0, \sigma^2)$. Determine the general form of the h-step BLP, $P_n X_{n+h}$, and show that its MSE is:
$$\sigma_n^2(h) \equiv \mathbb{E}(X_{n+h} - P_n X_{n+h})^2 = \frac{1 - \phi^{2h}}{1 - \phi^2}\sigma^2.$$

4. B&D Problem 5.9.

5. (B&D Problem 5.11) Given two observations $\{x_1, x_2\}$ from the causal AR(1) process $X_t = \phi X_{t-1} + Z_t$, where $Z_t \sim \text{WN}(0, \sigma^2)$, and assuming that $|x_1| \neq |x_2|$, find the MLEs of ϕ and σ^2.

6. S&S Problem 2.4.

7. S&S Problem 2.5.

8. (S&S Problem 3.38.) Consider the ARIMA model
$$X_t = \theta Z_{t-2} + Z_t, \qquad \text{where } Z_t \sim \text{WN}(0, \sigma^2).$$
   (a) Identify the model using the notation: $ARIMA(p, d, q) \times (P, D, Q)_s$.
   (b) Show that the series is invertible for $|\theta| < 1$, and find the cofficients in the representation:
$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}.$$
   (c) Develop equations for the h-step ahead forecast based on the infinite past, $\tilde{P}_n X_{n+h}$, and its variance.

Numerical work and simulations

9. (B&D Problem 5.3.) Consider the AR(2) process X_t satisfying:

$$X_t - \phi X_{t-1} - \phi^2 X_{t-2} = Z_t, \qquad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2).$$

   (a) For what values of ϕ is this a causal process?

   (b) The following sample moments were computed after observing $X_1, \ldots, X_{200}$:

$$\hat{\gamma}(0) = 6.06, \qquad \hat{\rho}(1) = 0.687.$$

   Find estimates of ϕ and σ^2 by solving the Yule–Walker equations. (If you find more than one solution, choose the one that is causal.)

10. B&D Problem 6.11(a).

Applications

11. B&D Problem 6.13.

12. S&S Problem 3.36.

13. S&S Problem 3.42.

14. S&S Problem 5.3.