STAT 5379 Assignment 1

(Covered in Homeworks 1–5)

Mathematical concepts and derivations

1. (B&D Problem 1.1) Let X and Y be two random variables with $\mathbb{E}(Y) = \mu$ and $\mathbb{E}Y^2 < \infty$.
   (a) Show that the constant c that minimizes $\mathbb{E}(Y - c)^2$ is $c = \mu$.
   (b) Deduce that the random variable $f(X)$ that minimizes:
   $$\mathbb{E}\left[(Y - f(X))^2 | X\right], \qquad \text{is} \qquad f(X) = \mathbb{E}[Y|X].$$
   (c) Deduce that the random variable $f(X)$ that minimizes:
   $$\mathbb{E}(Y - f(X))^2, \qquad \text{is also} \qquad f(X) = \mathbb{E}[Y|X].$$
2. (B&D Problem 1.4) Let $\{Z_t\} \sim$ IID N$(0, \sigma^2)$, and let a, b, and c be constants. Which, if any, of the following processes are stationary? For each stationary process specify the mean and autocovariance function.
   (a) $X_t = a + bZ_t + cZ_{t-2}$.
   (b) $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$.
   (c) $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$.
   (d) $X_t = a + bZ_0$.
   (e) $X_t = Z_0 \cos(ct)$.
   (f) $X_t = Z_t Z_{t-1}$.
3. (B&D Problem 1.7) Let $\{X_t\}$ and $\{Y_t\}$ be uncorrelated stationary sequences, i.e., X_r and Y_s are uncorrelated for every r and s. Show that $\{X_t + Y_t\}$ is stationary with autocovariance function equal to the sum of the autocovariance functions of $\{X_t\}$ and $\{Y_t\}$.
4. (B&D Problem 1.14) Show that the filter with coefficients $[a_{-2}, a_{-1}, a_0, a_1, a_2] = [-1, 4, 3, 4, -1]/9$ passes third-degree polynomials and eliminates seasonal components with period 3.
5. (B&D Problem 1.15b) Let $\{Y_t\}$ be a stationary process with mean zero, and let a and b be constants. If $X_t = (a + bt)s_t + Y_t$, where s_t is a seasonal component with period 12, show that $(1 - B^{12})^2 X_t$ is stationary, and express its autocovariance function in terms of that of $\{Y_t\}$.
6. S&S Problem 1.19.
7. (B&D Problem 2.14) Consider the process
   $$X_t = A\cos(\omega t) + B\cos(\omega t), \qquad t = 0, \pm1, \pm2, \ldots,$$
   where A and B are uncorrelated random variables with mean 0 and variance 1 and ω is a fixed frequency in the interval $[0, \pi]$.

(a) Find P_1X_2 and its mean squared error.
(b) Find P_2X_3 and its mean squared error.

8. B&D Problem 3.9.

9. S&S Problem 3.2.

Numerical work and simulations

10. S&S Problem 1.2.

11. S&S Problem 1.5.

12. S&S Problem 1.22.

13. (B&D Problem 2.12) Suppose that in a sample of size $n = 100$ from an MA(1) process with mean μ, $\theta = -0.6$, and $\sigma^2 = 1$ we obtain $\overline{x} = 0.157$. Construct an approximate 95% confidence interval for μ. Are the data compatible with the hypothesis that $\mu = 0$?

14. (B&D Problem 2.13a) Suppose that in a sample of size $n = 100$, we obtain $\hat{\rho}(1) = 0.438$ and $\hat{\rho}(2) = 0.145$. Assuming that the data were generated from an AR(1) model, construct approximate 95% confidence intervals for both $\rho(1)$ and $\rho(2)$. Based on these two confidence intervals, are the data consistent with an AR(1) model with $\phi = 0.8$?

15. (B&D Problem 3.1) Determine which of the following ARMA processes are causal and which of them are invertible. (In each case $\{Z_t\}$ denotes white noise.)

    (a) $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Zt$.
    (b) $X_t + 1.9X_{t-1} + 0.88X_{t-2} = Z_t + 0.2Z_{t-1} + 0.7Z_{t-2}$.
    (c) $X_t + 0.6X_{t-1} = Z_t + 1.2Z_{t-1}$.
    (d) $X_t + 1.8X_{t-1} + 0.81X_{t-2} = Z_t$.
    (e) $X_t + 1.6X_{t-1} = Z_t - 0.4Z_{t-1} + 0.04Z_{t-2}$.

    (Note: you can use the function "factor.wge" in R package `tswge` to get a nice summary table of the roots of the AR and MA polynomials, just be careful to reverse the signs of the MA coefficients because the package assumes the convention in the book by Woodward et al, 2017.)

16. B&D Problem 3.3.

17. S&S Problem 3.4.

18. (S&S Problem 3.7b) For the AR(2), $X_t - 0.4X_{t-1} - 0.45X_{t-2} = Z_t$, where $Z_t \sim \text{WN}(0, \sigma^2)$, use the results of Example 3.10 (the Yule-Walker equations) to determine a set of difference equations that can be used to find the ACF $\rho(h)$, for $h = 0, 1, \ldots$. Then plot the ACF values to lag 10 (use "ARMAacf" as a check on your answers).

19. (S&S Problem 3.9b) Generate $n = 100$ observations from the AR(2) model in the above problem. Compute the sample ACF and compare it to the theoretical values. Also compute the sample PACF and compare both the sample ACFs and PACFs with the general results predicted by Table 3.1 (S&S book).

Applications

20. S&S Problem 2.8.

21. S&S Problem 2.11.