STAT 5374 Hwk #5 ①

1. Let $\mathbf{x} = (x_1, x_2, x_3)^T$ and $\mathbf{y} = (y_1, y_2, y_3)^T$ be independent random vectors with $E(\mathbf{x}) = (2, 3, 4)^T$, $E(\mathbf{y}) = (2, 4, 6)^T$,

$$\text{var}(\mathbf{x}) = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 9 & 1 \\ 3 & 1 & 5 \end{pmatrix}, \quad \text{var}(\mathbf{y}) = \begin{pmatrix} 4 & -2 & 3 \\ -2 & 6 & 2 \\ 3 & 2 & 8 \end{pmatrix}.$$

In addition, let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Find

a. $\text{cov}(\mathbf{x}, \mathbf{y})$
b. $\text{var}(\mathbf{x} + \mathbf{y})$
c. $\text{cov}(\mathbf{Ax}, \mathbf{Bx})$
d. $E(\mathbf{Ax} + \mathbf{Ay})$
e. $\text{var}(\mathbf{Ax})$
f. $\text{corr}(\mathbf{Ax}, \mathbf{x})$
g. $\text{corr}(\mathbf{Ax}, \mathbf{Bx})$.

2. Problem 4.2 in our text.

3. Let $\mathbf{y} = (y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23})^T$ be the vector of weights of three pigs from each of two randomly selected litters at 3 months of age. A reasonable model for these variables states that

$$y_{ij} = G_i + e_{ij}, \quad i = 1, 2 \quad j = 1, 2, 3,$$

where y_{ij} represents the weight of the jth pig in the ith litter, $G_1, G_2, e_{11} e_{12}, \ldots, e_{23}$ are independent random variables, with $E(G_i) = \mu$, $\text{var}(G_i) = \sigma_G^2$, for each i, and $E(e_{ij}) = 0$, $\text{var}(e_{ij}) = \sigma_e^2$, for $(i, j) = (1, 1), \ldots, (2, 3)$. Here, G_i can be interpreted as the shared genetic effect for all pigs from the ith litter, while $e_{11}, \ldots, e_{23}$ are random deviations from G_i for the individual animals. Find $\text{var}(\mathbf{y})$, $\text{corr}(\mathbf{y})$, and $\text{var}(y_{..}) = \text{var}(\mathbf{j}_6^T \mathbf{y})$.

4. Let $\mathbf{x} = (x_1, x_2, x_3)^T$, $E(\mathbf{x}) = (2, 3, 4)^T$,

$$\text{var}(\mathbf{x}) = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix},$$

and $Q(\mathbf{x}) = \sum_{i=1}^{3} (x_i - \bar{x})^2$. Find $E\{Q(\mathbf{x})\}$.

5. Let $X_1, \ldots, X_n$ be uncorrelated random variables with equal means μ and $\text{var}(X_i) = \sigma_i^2$, $i = 1, \ldots, n$.

   a. For $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$, use property 5 on p. 65 of our notes to find $\text{var}(\bar{X})$.
   b. Find a constant K_n such that $Q = K_n \sum_{i=1}^n (X_i - \bar{X})^2$ is an unbiased estimator of $\text{var}(\bar{X})$. Thus even though that X_i's have unequal variances, Q is still an unbiased estimator of $\text{var}(\bar{X})$. $\bar{X}$ is not the linear unbiased estimator with the smallest variance, though. To see this, assume the σ_i^2's are known, take $n = 2$ and find the unbiased linear estimator with the smallest variance.

6. Let $\mathbf{x} = (x_1, x_2)^T$ have a bivariate normal distribution with parameters $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ for $-1 < \rho < 1$.

   a. Show that the density of $\mathbf{x}$ is

$$f_{\mathbf{x}}(\mathbf{x}) = \{(2\pi)^2\sigma_1^2\sigma_2^2(1-\rho^2)\}^{-1/2} \exp\left\{-\frac{1}{2}Q(\mathbf{x})/(1-\rho^2)\right\},$$

   for $Q(\mathbf{x}) = z_1^2 + z_2^2 - 2\rho z_1 z_2$ where $z_1 = (x_1 - \mu_1)/\sigma_1$, $z_2 = (x_2 - \mu_2)/\sigma_2$. (This is the bivariate normal density function.)
   b. Show that the conditional distribution of x_2 given $x_1 = c$, is

$$N\left(\mu_2 + \rho\sigma_2\left(\frac{c - \mu_1}{\sigma_1}\right), \sigma_2^2(1-\rho^2)\right).$$

   Thus, for example, if the heights in inches of fathers (x_1) and sons (x_2) have a bivariate normal distribution with means $\mu_1 = 69, \mu_2 = 70, \sigma_1 = 2, \sigma_2 = 3, \rho = .4$, then the conditional distribution of a son's height given that his father is 73 inches tall (2 s.d.'s above the average) is $N(72.4, 7.56)$ (0.8 s.d.'s above average with variance $0.84\sigma_2^2$).

7. Problem 4.17 in our text.