STAT 5374 Hwk #2

1. Let $\mathbf{x}_1 = \mathbf{j}_4$, $\mathbf{x}_2 = (4,1,3,4)^T$, $\mathbf{y} = (1,9,5,5)^T$. Let $V = \mathcal{L}(\mathbf{x}_1, \mathbf{x}_2)$.

   a. Find $\hat{\mathbf{y}} = p(\mathbf{y}|V)$ and $\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}}$.
   b. Find $\hat{\mathbf{y}}_1 = p(\mathbf{y}|\mathbf{x}_1)$ and $\hat{\mathbf{y}}_2 = p(\mathbf{y}|\mathbf{x}_2)$ and show that $\hat{\mathbf{y}} \neq \hat{\mathbf{y}}_1 + \hat{\mathbf{y}}_2$.
   c. Verify that $\mathbf{e} \perp V$.
   d. Find $||\mathbf{y}||^2$, $||\hat{\mathbf{y}}||^2$, $||\mathbf{e}||^2$, and verify that the Pythagorean Theorem holds. Compute $||\hat{\mathbf{y}}||^2$ directly from $\hat{\mathbf{y}}$ and also by using the formula $||\hat{\mathbf{y}}||^2 = \mathbf{y}^T\mathbf{P}\mathbf{y}$ where $\mathbf{P}$ is the projection matrix onto V.
   e. Use Gram-Schmidt orthogonalization to find four mutually orthogonal vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$ such that $V = \mathcal{L}(\mathbf{v}_1, \mathbf{v}_2)$. *Hint:* You can choose $\mathbf{x}_3$ and $\mathbf{x}_4$ arbitrarily, as long as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ are LIN.

2. (Simple linear regression.) Let $\mathbf{y} = (y_1, \ldots, y_n)^T$, $\mathbf{x} = (x_1, \ldots, x_n)^T$, and $V = \mathcal{L}(\mathbf{j}_n, \mathbf{x})$.

   a. Use Gram-Schmidt orthogonalization on the vectors $\mathbf{j}_n$, $\mathbf{x}$ (in this order) to find orthogonal vectors $\mathbf{j}_n, \mathbf{x}^*$ spanning V. Express $\mathbf{x}^*$ in terms of $\mathbf{j}_n$ and $\mathbf{x}$, then find b_0, b_1 such that $\hat{\mathbf{y}} = b_0\mathbf{j}_n + b_1\mathbf{x}$. To simplify the notation, let $\mathbf{y}^* = \mathbf{y} - p(\mathbf{y}|\mathbf{j}_n) = \mathbf{y} - \bar{y}\mathbf{j}_n$,

$$S_{xy} = \langle \mathbf{x}^*, \mathbf{y}^* \rangle = \langle \mathbf{x}^*, \mathbf{y} \rangle = \sum_i (x_i - \bar{x})(y_i - \bar{y}) = \sum_i (x_i - \bar{x})y_i = \sum_i x_i y_i - n\bar{x}\bar{y},$$

$$S_{xx} = \langle \mathbf{x}^*, \mathbf{x}^* \rangle = \sum_i (x_i - \bar{x})^2 = \sum_i (x_i - \bar{x})x_i = \sum_i x_i^2 - n\bar{x}^2,$$

$$S_{yy} = \langle \mathbf{y}^*, \mathbf{y}^* \rangle = \sum_i (y_i - \bar{y})^2.$$

   b. Suppose $\hat{\mathbf{y}} = p(\mathbf{y}|V) = a_0\mathbf{j}_n + a_1\mathbf{x}^*$. Find formulas for a_1 and A_0 in terms of $\bar{y}$, S_{xy}, and S_{xx}.

   c. Express $\mathbf{x}^*$ in terms of $\mathbf{j}_n$ and $\mathbf{x}$, and use this to determine formulas for b_0 and b_1 so that $\hat{\mathbf{y}} = b_0\mathbf{j}_n + b_1\mathbf{x}$.
   d. Express $||\hat{\mathbf{y}}||^2$ and $||\mathbf{y} - \hat{\mathbf{y}}||^2$ in terms of S_{xy}, S_{xx} and S_{yy}.
   e. Use the formula $\mathbf{b} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$ for $\mathbf{b} = (b_0, b_1)^T$, and verify that this gives the same answer as in (c).
   f. for $\mathbf{y} = (2,6,8,8)^T$, $\mathbf{x} = (0,1,2,3)^T$ find $a_0, a_1, \hat{\mathbf{y}}, b_0, b_1, ||\mathbf{y}||^2, ||\hat{\mathbf{y}}||^2, ||\mathbf{y} - \hat{\mathbf{y}}||^2$. Verify that $||\hat{\mathbf{y}}||^2 = b_0\langle \mathbf{y}, \mathbf{j}_4 \rangle + b_1\langle \mathbf{y}, \mathbf{x} \rangle$ and that $(\mathbf{y} - \hat{\mathbf{y}}) \perp V$.

3. Let $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be a basis of a subspace $V \subset \mathcal{R}^n$. Suppose that $p(\mathbf{y}|V) = \sum_{j=1}^k p(\mathbf{y}|\mathbf{x}_j)$ for every vector $\mathbf{y} \in \mathcal{R}^n$. Prove that $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are mutually orthogonal. *Hint:* Consider the vector $\mathbf{y} = \mathbf{x}_i$ for each i.

4. Show that for $\mathbf{W}_{n\times k} = \mathbf{X}_{n\times k}\mathbf{B}_{k\times k}$ with $\mathbf{B}$ nonsingular and $\mathbf{X}$ of full rank, $\mathbf{P} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$ remains unchanged if $\mathbf{X}$ is replaced by $\mathbf{W}$. Thus $\mathbf{P}$ is a function of the subspace spanned by the columns of $\mathbf{X}$, not of the particular basis chosen for this subspace (we can change $\mathbf{X}$ without affecting $\mathbf{P}$ as long as we haven't changed $C(\mathbf{X})$).

5. For each subspace V of $\mathcal{R}^3$ give the corresponding projection matrix $\mathbf{P}$. In each case verify that $\mathbf{P}$ is symmetric and idempotent.

   a. $V = \mathcal{L}(\mathbf{x})$ where $\mathbf{x} = (2,-1,-1)^T$.
   b. $V = \mathcal{L}(\mathbf{x}_1, \mathbf{x}_2)$ where $\mathbf{x}_1 = (1,1,1)^T$, and $\mathbf{x}_2 = (1,-1,0)^T$.

6. For the subspace $V = \mathcal{L}(\mathbf{j}_n, \mathbf{x})$ of problem 2, what is $\mathbf{P}_V$? (Note that $V = \mathcal{L}(\mathbf{j}_n, \mathbf{x}^*)$, also.) What is $\mathbf{P}_{V^\perp}$?