STAT 5374 Hwk #1.

1. Consider the following vectors in $\mathcal{R}^5$: $\mathbf{x} = (2,1,1,1,4)^T$, and $\mathbf{y} = (3,6,1,5,7)^T$.

   a. Find $\langle \mathbf{x}, \mathbf{y} \rangle$, $||\mathbf{x}||^2$, $||\mathbf{y}||^2$, $\hat{\mathbf{y}} = p(\mathbf{y}|\mathbf{x})$, and $\mathbf{y} - \hat{\mathbf{y}}$. Show that $\mathbf{x} \perp (\mathbf{y} - \hat{\mathbf{y}})$, and $||\mathbf{y}||^2 = ||\hat{\mathbf{y}}||^2 + ||\mathbf{y} - \hat{\mathbf{y}}||^2$.
   b. Let $\mathbf{w} = (-1,2,4,-4,0)^T$ and $\mathbf{z} = 3\mathbf{x} + 2\mathbf{w}$. Show that $\langle \mathbf{w}, \mathbf{x} \rangle = 0$ and that $||\mathbf{z}||^2 = 9||\mathbf{x}||^2 + 4||\mathbf{w}||^2$. (Why must this be true?)
   c. Let $A_1 = \{1,2\}, A_2 = \{3,4\}, A_3 = \{5\}$. Find $p(\mathbf{y}|\mathbf{i}_{A_i})$, $i = 1,2,3$.

2. Is projection a linear transformation in the sense that $p(c\mathbf{y}|\mathbf{x}) = cp(\mathbf{y}|\mathbf{x})$ for any real number c? Prove or disprove. What is the relationship between $p(\mathbf{y}|\mathbf{x})$ and $p(\mathbf{y}|c\mathbf{x})$ for $c \neq 0$?

3. Suppose $||\mathbf{x}||^2 > 0$. Use calculus to prove that $||\mathbf{y} - b\mathbf{x}||^2$ is minimum for $b = \mathbf{y}^T\mathbf{x}/||\mathbf{x}||^2$. (That is, provide a calculus-based proof of the Theorem on p. 22 of the notes as an alternative to the one provided there.)

4. Here we will consider subspaces of $\mathcal{R}^4$. Let $\mathbf{x}_1 = (1,1,1,0)^T$, $\mathbf{x}_2 = (1,1,0,0)^T$, $\mathbf{x}_3 = (1,0,0,1)^T$ and $\mathbf{x}_4 = (10,3,0,7)^T$. Let $V_2 = \mathcal{L}(\mathbf{x}_1, \mathbf{x}_2)$, $V_3 = \mathcal{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ and $V_4 = \mathcal{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$.

   a. Find matrices $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ whose column spaces are equal to V_2, V_3, V_4, respectively.
   b. Find the dimensions of V_2 and V_3 (i.e., the ranks of $\mathbf{A}_2$ and $\mathbf{A}_3$).
   c. Find bases for V_2 and V_3 that contain vectors with as many zeros as possible.
   d. Give a vector $\mathbf{z} \neq \mathbf{0}$ that is orthogonal to all vectors in V_3.
   e. $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and $\mathbf{z}$ are linearly independent. Therefore, $\mathbf{x}_4$ is expressible in the form $\sum_{i=1}^3 b_i\mathbf{x}_i + c\mathbf{z}$ (this is true because one can form at most d linearly independent vectors in a vector space of dimension d). Show that $c = 0$ and hence that $\mathbf{x}_4 \in V_3$, by determining $\mathbf{x}_4^T\mathbf{z}$. What is $\dim(V_4)$ (the dimension of V_4)?
   f. Give a simple description of V_3 in words.

5. Consider $\mathcal{R}^6$, Euclidean 6-space, but let's index a vector $\mathbf{y}$ in this space as $\mathbf{y} = (y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{31})^T$. Think of this indexing scheme as representing data y_{ij} from 3 treatment groups, with 3, 2, and 1 replicates (indexed by j) in groups 1, 2, and 3, respectively (indexed by i). That is, y_{ij} represents the j^{th} response in the i^{th} group where $i = 1,2,3$ and $j = 1,\ldots,n_i$, $n_1 = 3$, $n_2 = 2$, $n_3 = 1$. Let $\mathbf{a}_i$ be the indicator for the i^{th} treatment group (e.g., $a_1 = (1,1,1,0,0,0)^T$, $a_2 = (0,0,0,1,1,0)^T$, etc.). Let $V = \mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$.

   a. For $\mathbf{y} = (2,1,0,7,9,3)^T$ find $\hat{\mathbf{y}} = p(\mathbf{y}|V)$, $\mathbf{y} - \hat{\mathbf{y}}$, $||\mathbf{y}||^2$, $||\hat{\mathbf{y}}||^2$, $||\mathbf{y} - \hat{\mathbf{y}}||^2$.
   b. Give a general non-matrix formula for $\hat{\mathbf{y}} = p(\mathbf{y}|V)$ for any $\mathbf{y}$.

6. Problem 2.23 from the textbook.

7. Problem 2.24 " " " .