Complex Variable I A. Yu. Solynin

HW #1 MATH 5320 - 1 Fall 2012 08.27 Due September 12, 2012

(1) Perform the required calculations and express your answer in the algebraic (cartesian) form $a + ib$.
  (a) $(1+i)^6$
  (b) $\Im\left(\frac{1+2i}{3-4i}\right)$
  (c) $\Re\left((1-i)^{-2}\right)$
  (d) $\Re\left((x+iy)(x-iy)\right)$
  (e) $\sqrt{i-1}$
  (f) $\sqrt[4]{i}$

(2) Solve the given equations.
  (a) $z^3 - 7z^2 + 6z - 10 = 0$
  (b) $z^4 - 2z^2 + 5 = 0$

(3) Identify and sketch the set of points satisfying:
  (a) $|z - (1+i)| = 1$
  (b) $2 < |z| < 5$
  (c) $0 < \Im z < \pi$
  (d) $|\Re z| + |\Im z| \le 1$
  (e) $|z-2| + |z+2| = 6$
  (f) $|z-1| = |z+i|$
  (g) $\Re(z+1) = |z-1|$
  (h) $\{w = z^2 : \Re z = 1\}$

(4) Prove the following:
  (a) $|\Re z| \le |z|$
  (b) $|z+w|^2 = |z|^2 + |w|^2 + 2\Re(z\bar{w})$
  (c) $|z+w| \le |z| + |w|$ (Triangle inequality)

(5) Perform the required calculations and express your answer in both algebraic (cartesian) and polar forms.
  (a) $\sqrt{i-1}$
  (b) $\sqrt[4]{i}$

(6) Identify and sketch the set of points satisfying:
  (a) $|\arg z| < \pi/4$
  (b) $\{z : |z| < 4, \frac{\pi}{3} < \arg z < \frac{3\pi}{2}\}$
  (c) $|z| = \arg z$
  (d) For a fixed $b > 0$, sketch the curve $\{e^{i\theta} + be^{-i\theta} : 0 \le \theta \le 2\pi\}$. Hint. Differentiate between the cases $0 < b < 1$, $b = 1$, and $b > 0$.

(7) For $n > 1$, prove that
  (a) $1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$, $z \ne 1$. (Does this formula look familiar?)
  (b) $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\theta/2}$
  (c) The sum of the n-th roots of 1 equals 0.

(8) Prove the following:
  (a) If the point P on the sphere corresponds to z under the stereographic projection, then the antipodal point $-P$ on the sphere corresponds to $-\frac{1}{\bar{z}}$.
  (b) A rotation of the sphere of 180^o about the X-axis corresponds under stereographic projection to the inversion $z \mapsto \frac{1}{z}$ of $\mathbb{C}$.
  (c) $\rho(z, \infty) = \frac{2}{\sqrt{1+|z|^2}}$