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Travelling Wave Solutions of the Codazzi-Betchov-Da Rios Evolution Equations

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Energy Functionals

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- We are going to consider energy functionals acting on $\Omega_{p_o p_1}$ of the following form

$$\Theta(\gamma) = \int_\gamma \mathcal{F}(\kappa) + \mu\tau + \lambda = \int_0^L \left(\mathcal{F}(\kappa)(s) + \mu\tau(s) + \lambda\right) ds\,,$$

where $\mathcal{F}(u)$ is a $C^\infty(\mathbb{R})$ function and $\mu, \lambda \in \mathbb{R}$.

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  where $\mathcal{F}(u)$ is a $C^\infty(\mathbb{R})$ function and $\mu, \lambda \in \mathbb{R}$.
- A version of the Lagrange multipliers allows us to interpret this variational problem as the minimization of the curvature energy $\int_\gamma \mathcal{F}(\kappa)$ subject to two constraints: fixed length and fixed total torsion.

Reduction Theorem

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Reduction Theorem [2]

A critical point of Θ must lie in a 3-dimensional totally geodesic submanifold of $M_r^n(\rho)$.

Thus, we are interested in studying critical curves in pseudo-Riemannian 3-space forms, $M_r^3(\rho)$.

Euler-Lagrange Equations

The Euler-Lagrange equations for the curvature energy functional $\Theta(\gamma) = \int_\gamma \mathcal{F}(\kappa) + \mu\tau + \lambda$, acting on $\Omega_{p_o p_1}$ can be written as

$$\begin{aligned} \mu\kappa\tau &= \kappa(\mathcal{F} + \lambda) - \dot{\mathcal{F}}(\kappa^2 - \varepsilon_1\varepsilon_3\tau^2 + \varepsilon_2\rho) - \varepsilon_1\varepsilon_2\dot{\mathcal{F}}_{ss}\,, \\ \mu\kappa_s &= -2\varepsilon_1\varepsilon_3\tau\dot{\mathcal{F}}_s - \varepsilon_1\varepsilon_3\tau_s\dot{\mathcal{F}}\,. \end{aligned}$$

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Generalized Kirchhoff Centerlines ([1], [2])

Curves whose curvature and torsion satisfy above equations will be called generalized Kirchhoff centerlines.

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Thus, under suitable boundary conditions, generalized Kirchhoff centerlines are critical curves of our energy functionals.

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$$W(v)(\bar{t},0) = W(\kappa)(\bar{t},0) = W(\tau)(\bar{t},0) = 0\,,$$

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Characterization of Centerlines ([1], [2])

The vector field $\mathcal{I} = \varepsilon_1\varepsilon_3\mu T + \dot{\mathcal{F}}B$ is a Killing vector field along γ, if and only if, γ is a generalized Kirchhoff centerline.

Binormal Evolution Surfaces

1. Evolution of Curves

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3. Fundamental Equations

Evolution of Curves

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Properties ([1], [2])

1. This is a length-preserving evolution.
2. The initial condition $\gamma(s) = x(s,0)$ evolves by the binormal flow, $x_t = \dot{P}(\kappa)B$, where $\dot{P} = \varepsilon_2\varepsilon_3\kappa f(\kappa)$.

BINORMAL EVOLUTION SURFACES ([1], [2])

The corresponding immersed surface (U, x) in $M_r^3(\rho)$ swept out by $\gamma(s)$ will be denoted S_γ and called a binormal evolution surface with initial condition γ and velocity $\dot{P}$.

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3. The second fundamental form of S_γ

$$\varepsilon_2 h = -\kappa ds^2 + 2\tau \dot{P} ds\, dt + \varepsilon_2 \dot{P}^2 h_{22} dt^2 .$$

Fundamental Equations

The term h_{22} is defined by

$$h_{22} = \langle \widetilde{\nabla}_{e_2} e_2, e_3 \rangle = \frac{1}{\kappa}\{\varepsilon_3 \frac{\dot{P}_{ss}}{\dot{P}} - \varepsilon_2 \tau^2 + \varepsilon_1 \varepsilon_3 \rho\}.$$

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Using this, we see that the Gauss-Codazzi equations boil down to

$$\begin{aligned} \kappa_t &= -2\dot{P}_s \tau - \tau_s \dot{P}\,, \\ \varepsilon_3 \tau_t &= \left(\frac{1}{\kappa}\left(\varepsilon_2 \dot{P}_{ss} + \varepsilon_1 \dot{P}(\kappa^2 - \varepsilon_1 \varepsilon_3 \tau^2 + \varepsilon_2 \rho) - \varepsilon_1 \kappa P\right)\right)_s \,. \end{aligned}$$

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Fundamental Theorem of Submanifolds ([1], [2])

For any pair of functions $\kappa(s,t)$, $\tau(s,t)$ satisfying the Gauss-Codazzi equations, there exists an isometric immersion $x : U \to M_r^3(\rho)$ foliated by a family of geodesics $\gamma^t(s) = x(s,t)$ evolving by the binormal flow.

Travelling Wave Solutions

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2. Foliations of Binormal Evolution Surfaces

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3. Applications

Travelling Wave Solutions

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Travelling Wave Solutions of Gauss-Codazzi Equations ([1], [2])

They correspond to the curvature and torsion of generalized Kirchhoff centerlines.Moreover, generalized Kirchhoff centerlines evolve following the binormal flow by isometries of $M_r^3(\rho)$ and slippery.

Foliations of BES

Thus, we have

1. A geodesic foliation $\mathcal{F} = \{\gamma^t\}_{t\in\mathbb{R}}$ of S_γ.

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THEOREM [2]

Consider the pseudo-Riemannian manifold $(B \times F, g)$, whose canonical foliations $\mathcal{F}_B$ and $\mathcal{F}_F$ are orthogonal everywhere. Then, the metric g is a warped product metric, if and only if, $\mathcal{F}_B$ is a totally geodesic foliation and $\mathcal{F}_F$ is a spherical foliation.

Applications

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1. Hasimoto Surfaces
    - O. J. Garay, A. Pámpano and C. Woo, Hypersurface constrained elasticae in Lorentzian space forms, *Advances in Mathematical Physics* 2015, 2015, Article ID 458178, 13 pp.
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3. Constant Mean Curvature BES
    - W. Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativittstheorie I: Elementare Differentialgeometrie*, Springer, Berlin, 1930.
    - O. J. Garay and A. Pámpano, On a Blaschke's variational problem. In preparation, 2017.

References

1. O. J. Garay, A. Pámpano, Binormal Evolution of Curves with Prescribed Velocity, *WSEAS transactions on fluid mechanics* 11, 2016, pp. 112-120.
2. O. J. Garay and A. Pámpano, Travelling Wave Solutions of the Extended Codazzi-Betchov-Da Rios Evolution Equations, In preparation, 2017.

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