Minimal Surfaces Bounded By Elastic Curves

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Abstract. We study equilibrium compact surfaces with boundary for an energy which is a linear combination of the Willmore energy and a second term which measures the bending of the boundary, focusing our attention mainly on minimal surfaces. In this case, the original problem reduces to the Plateau problem for fixed boundary elastic curves, with some topological restrictions.

INTRODUCTION

These brief notes are a printed version of the talk given by the author at the *18th International Conference of Numerical Analysis and Applied Mathematics (Elastic Curves and Surfaces with Applications and Numerical Representations)*. The purpose of the talk was to offer a partial announcement of some results included in the joint work with Prof. Bennett Palmer, [6]. Interested readers are referred to this work for a complete and more general treatment.

The theory of elasticity is a classical subject which led to the earliest developments of *Calculus of Variations*, [4]. In 1691, the classical problem of determining the shape of an ideal elastic rod which is being held bent by external forces and moments acting at its ends was first formulated by J. Bernoulli. Following a model of D. Bernoulli, this rod should bend along an *elastic curve*, i.e. a critical curve for the potential energy of strain

$$\mathcal{E}[C] := \int_C \left(\kappa^2 + \lambda\right) ds\,.$$

Using this formulation, L. Euler described the possible qualitative types for planar rod configurations, [1].

Much later, in 1811, S. Germain suggested to measure the free energy controlling the physical system associated with an elastic plate by an integral over the plate surface. One of the simplest functionals to be considered is the two dimensional counterpart of the bending energy $\mathcal{E}[C]$,

$$\mathcal{F}[X] := \int_\Sigma H^2\, d\Sigma\,.$$

Although this variational problem was already studied by Blaschke's school in the 1920's, it is nowadays known as the *Willmore energy* due to the contributions of T. J. Willmore, [7].

Since then, in order to understand the geometry of naturally occurring surfaces, an ever increasing variety of potential energy functionals have been studied. In these notes, we consider a combination of the Willmore energy of surfaces and the elastic energy of curves for compact surfaces with boundary.

EULER-WILLMORE VARIATIONAL PROBLEM

Let Σ be a compact, connected surface with boundary and consider the immersion of Σ in the Euclidean 3-space, $\mathbb{R}^3$,

$$X : \Sigma \to \mathbb{R}^3\,.$$

We assume that $X(\Sigma)$ is an oriented surface of class C^4 embedded in $\mathbb{R}^3$ with sufficiently smooth boundary, $\partial\Sigma$ (the boundary $\partial\Sigma$ is always considered as being positively oriented). We fix ν to be a *unit normal* vector field along Σ, and denote by H the *mean curvature* of the immersion.

The connected components of the boundary of Σ, $\partial\Sigma$, will be represented by arc-length parameterized curves C. For a sufficiently smooth curve $C : I \to \mathbb{R}^3$, we denote by $s \in I = [0, \mathcal{L}]$ the arc-length parameter of C, where $\mathcal{L}$ stands for its *length*. Then, if $()'$ represents the derivative with respect to the arc-length, the vector field $T(s) := C'(s)$ is the *unit tangent* to C. Moreover, the *(Frenet) curvature* of C, κ, is defined by $\kappa(s) := \|T'(s)\| \geq 0$.

For an immersion $X : \Sigma \to \mathbb{R}^3$, the *Euler-Willmore functional* is the potential energy ($\mathcal{W} \equiv \mathcal{W}_{a,\alpha,\beta}$)

$$\mathcal{W}[X] := a \int_\Sigma H^2 \, d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \beta\right) ds \,, \tag{1}$$

where $a > 0$, $\alpha > 0$ and $\beta \in \mathbb{R}$. The parameter a is the *bending rigidity* of the surface, α represents the *flexural rigidity* of the boundary and β is the *line tension* which can be interpreted as a Lagrange multiplier fixing the length of the boundary. For convenience, we assume that all connected components of the boundary, $\partial\Sigma$, are made of the same material, so that the parameters α and β are the same constants for all boundary components.

To compute the first variation of the potential energy $\mathcal{W}[X]$, (1), we first introduce the *Darboux frame*. The Darboux frame of $\partial\Sigma$ is the orthonormal frame $\{n, T, \nu\}$, where $n := T \times \nu$ denotes the *conormal* of the boundary. The derivative of this frame with respect to the arc-length parameter s is given by

$$\begin{aligned} n' &= -\kappa_g T + \tau_g \nu \,, \\ T' &= \kappa_g n + \kappa_n \nu \,, \\ \nu' &= -\tau_g n - \kappa_n T \,, \end{aligned}$$

where the functions involved, κ_g, κ_n and τ_g, are, respectively, the *geodesic curvature*, the *normal curvature* and the *geodesic torsion*.

Consider arbitrary variations of the immersion $X : \Sigma \to \mathbb{R}^3$, i.e. $X + \epsilon\delta X + O(\epsilon^2)$. Then, by standard arguments of *Calculus of Variations* involving integration by parts, we obtain the *Euler-Lagrange equations*. By considering compactly supported variations, on Σ,

$$\Delta H + 2H\left(H^2 - K\right) = 0 \,, \tag{2}$$

holds. At the same time, from the first variation formula, we also get the following boundary conditions

$$H = 0 \,, \tag{3}$$

$$J' \cdot \nu - a\partial_n H = 0 \,, \tag{4}$$

$$J' \cdot n = 0 \,, \tag{5}$$

where J' is the derivative with respect to the arc-length parameter of the vector field J defined along $\partial\Sigma$ as

$$J := 2\alpha T'' + \left(3\alpha\kappa^2 - \beta\right) T \,.$$

Equations (3) and (4) come from normal variations, while (5) is deduced by considering variations tangent to the surface.

For a rescaling of the immersion $X : \Sigma \to \mathbb{R}^3$ for $\sigma > 0$, i.e. $X \to \sigma X$, the Willmore energy remains invariant, the length of the boundary rescales linearly, while the elastic boundary term rescales like σ^{-1}. Therefore, if $X : \Sigma \to \mathbb{R}^3$ is a critical immersion for the potential energy $\mathcal{W}[X]$, (1), the following relation holds:

$$\beta = \frac{\alpha}{\mathcal{L}[X|_{\partial\Sigma}]} \oint_{\partial\Sigma} \kappa^2 \, ds \,,$$

where $\mathcal{L}[X|_{\partial\Sigma}]$ denotes the length of the boundary $\partial\Sigma$. In particular, the line tension, β, must be positive. From now on, we assume that $\beta > 0$ holds.

EQUILIBRIUM CONFIGURATIONS WITH CONSTANT MEAN CURVATURE

Throughout this section, let us consider that $X : \Sigma \to \mathbb{R}^3$ is a critical immersion for the potential energy $\mathcal{W}[X]$, (1), with constant mean curvature. Clearly, from the Euler-Lagrange equation (3), we conclude that $X(\Sigma)$ is a *minimal*

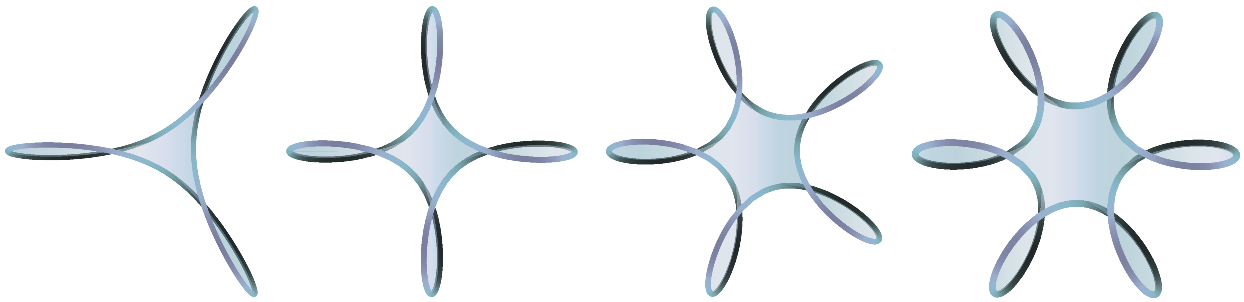

FIGURE 1. Minimal surfaces of disc type spanned by elastic curves of type $G(q, 1)$ for $q = 3, 4, 5$ and 6. These configurations are critical for $\mathcal{W}[X]$, (1).

surface, i.e. $H \equiv 0$ holds on Σ. In this case, both (2) and (3) are satisfied, while (4) and (5) become the Euler-Lagrange equations of the classical elastic energy in $\mathbb{R}^3$,

$$\mathcal{E}[C] := \int_C \left(\kappa^2 + \lambda\right) ds \,, \tag{6}$$

with $\lambda := \beta/\alpha > 0$. That is, if $X : \Sigma \to \mathbb{R}^3$ is a minimal critical immersion for $\mathcal{W}[X]$, (1), the boundary components are closed and simple critical curves for $\mathcal{E}[C]$, (6), i.e. closed and simple elastic curves.

Among planar curves, the only closed critical curves for $\mathcal{E}[C]$, (6), are circles of radii $r_o = \sqrt{\alpha/\beta}$ and elastic figure-eights, the former being the only embedded ones. On the other hand, closed elastic curves in $\mathbb{R}^3$ were studied in [2], proving that there exist infinitely many embedded closed non planar elastic curves lying on rotational tori. These non planar elastic curves represent (q, p)-torus knots for $0 < 2p < q$, denoted here by $G(q, p)$. The parameters q and p have a geometric meaning. Indeed, p represents the number of rounds around the z-axis (after rigid motions) that the critical curve does in order to close, while q is the number of periods of the curvature needed to close. Closed planar elastic curves can be included in this family of (q, p)-torus knots by considering circles for $p = 0$ and elastic figure-eights for $q = 2p$. The deformation of the elastic circle into the elastic figure-eight through the family of non planar elastic curves can be visualized in: *https://www.youtube.com/watch?v=49CeK8g1RAo*. (This video also includes a counter of closed elastic curves.)

As a consequence, the original problem of obtaining critical immersions $X : \Sigma \to \mathbb{R}^3$ for $\mathcal{W}[X]$, (1), with $H \equiv 0$, reduces to seeking minimal surfaces whose boundary is composed by fixed closed and simple elastic curves (which represent torus knots of type $G(q, p)$). In the particular case that $\Sigma \cong D$ is a topological disc, this is the classical *Plateau problem* where the boundary is a fixed elastic curve.

In what follows we study separately two of the simplest topological types: disc type surfaces and annuli.

Disc Type Surfaces

Let $\Sigma \cong D$ be a topological disc and consider a minimal immersion $X : D \to \mathbb{R}^3$ critical for $\mathcal{W}[X]$, (1). As mentioned above, the boundary $X(\partial D)$ is the image of a closed and simple elastic curve, C.

First, if C is an elastic circle (i.e. its radius is $r_o = \sqrt{\alpha/\beta}$), then $X(D)$ is a planar disc bounded by C. On the other hand, if the elastic curve C is non planar, then it represents a torus knot $G(q, p)$ with $0 < 2p < q$. Therefore, in the latter case, $X(D)$ is a *Seifert surface*. We say that an orientable, connected surface that has as its boundary an oriented knot (or link) is a Seifert surface, [5]. The *genus* of a knot is defined as the minimum of the genus of any of its Seifert surfaces. In our case, the Seifert surface is a disc, so it has genus zero. Hence, the genus of the torus knot $G(q, p)$ with $0 < 2p < q$ is also zero. The genus of torus knots can be computed explicitly (for details, see [5]) and so we conclude that $p = 1$ and $q > 2$ must hold.

In conclusion, if $X : D \to \mathbb{R}^3$ is a minimal critical immersion for $\mathcal{W}[X]$, (1), of a topological disc, then either $X(D)$ is a planar disc bounded by a circle of radius $r_o = \sqrt{\alpha/\beta}$, or its boundary is a closed and simple non planar elastic curve representing a torus knot of type $G(q, 1)$ for $q > 2$. For fixed $q > 2$, one can try to use numerical algorithms to solve the Plateau problem for the fixed elastic curve of type $G(q, 1)$. In Figure 1, we show some minimal embedded disc type surfaces critical for $\mathcal{W}[X]$, (1). These surfaces have been obtained numerically implementing in *Wolfram Mathematica* an algorithm based on the mean curvature flow.

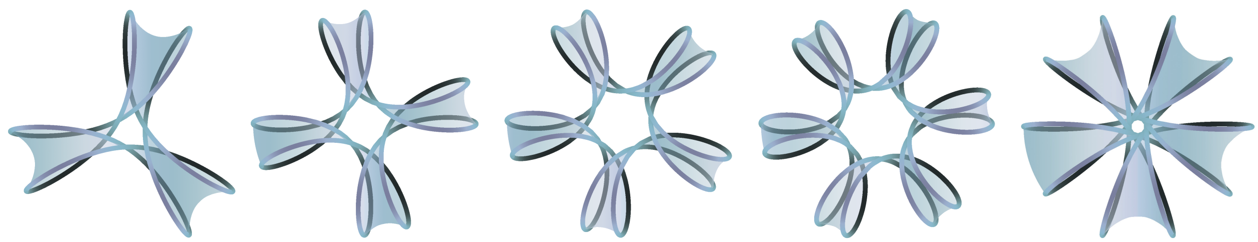

FIGURE 2. Minimal annuli with boundary the union of congruent elastic curves for different torus knots $G(q, p)$. From left to right: $G(3, 1)$, $G(4, 1)$, $G(5, 1)$, $G(6, 1)$ and $G(5, 2)$. These domains are critical for $\mathcal{W}[X]$, (1).

Topological Annuli

Assume now that $\Sigma \cong A$ is a topological annulus. If we seek minimal immersions critical for $\mathcal{W}[X]$, (1), then both boundary components are closed and simple elastic curves representing torus knots of type $G(q, p)$ with $0 \leq 2p < q$ (recall that the elastic circle is included here as $p = 0$).

In this setting, by cobordism theory (for details, see [3]), we obtain that both connected components of the boundary must be torus knots of the same type. The simplest case is when both of them are circles of radii $r_o = \sqrt{\alpha/\beta}$. An example of this minimal annular surface is a suitable symmetric domain in a catenoid. We point out here that there are also non axially symmetric minimal domains critical for $\mathcal{W}[X]$, (1). For instance, if we fix two circles of radii $r_o = \sqrt{\alpha/\beta}$ in two horizontal planes closed enough one from another, such that the circle's centers are not vertically aligned and search for the minimal surface whose boundary is the union of these circles, we get *Riemann's minimal examples*, which are non rotational minimal surfaces critical for $\mathcal{W}[X]$, (1).

However, if the boundary components are not circles, in order to produce examples, we can proceed as follows. Fix C_1 to be a closed and simple non planar elastic curve of type $G(q, p)$ with $0 < 2p < q$. This elastic curve C_1 lies on a torus of revolution, hence, a small enough suitable rotation of C_1 will give us a congruent copy, C_2, which also lies in the same torus. Next, we search for a minimal surface with boundary $C_1 \cup C_2$. Numerically, one way of doing this is to consider the piece of the rotational torus between C_1 and C_2 as the initial condition and then apply the mean curvature flow. Implementing this algorithm in *Wolfram Mathematica*, we have obtained some minimal annuli bounded by elastic curves representing different torus knots $G(q, p)$ (see Figure 2).

ABSOLUTE MINIMIZERS

In this last section, we are going to approach the problem of minimizing the potential energy $\mathcal{W}[X]$, (1). Following [6], for any immersion $X : \Sigma \to \mathbb{R}^3$ of a compact surface Σ with n boundary components, $\partial\Sigma \equiv \cup_{i=1}^n C_i$, we have that

$$\mathcal{W}[X] \geq \mathcal{W}_n := 4\pi\, n \sqrt{\alpha\beta}\;.$$

Equality in above estimate holds if and only if $X(\Sigma)$ is a minimal surface bounded by n circles of radii $r_o = \sqrt{\alpha/\beta}$, i.e. elastic circles.

For a topological disc ($\Sigma \cong D$ and $n = 1$) the minimum $\mathcal{W}_1$ is attained if and only if $X(D)$ is a planar disc bounded by a circle of radius $r_o = \sqrt{\alpha/\beta}$. However, for different topological types, there are multiple domains attaining the minima $\mathcal{W}_n$. For instance, for a topological annulus ($\Sigma \cong A$ and $n = 2$) all previous critical examples bounded by circles (suitable domains in the catenoid and in Riemann's minimal examples) attain the minimum $\mathcal{W}_2$.

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