A New Characterization of Biconservative Surfaces in 3-Space Forms

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From now on we will look for proper (non-CMC) biconservative surfaces in $N^3(\rho)$.

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- They are Weingarten surfaces ($\mathcal{W}(H, K) = 0$).
- They are linear Weingarten surfaces, i.e. **(Fu & Li, 2013)**

$$3\kappa_1 + \kappa_2 = 0 \,,$$

where $\kappa_1 = -\kappa$.

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Theorem (López & —, 2020)

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$$\mathbf{\Theta}_\mu(\gamma) = \int_\gamma (\kappa - \mu)^n$$

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- Biconservative case: $\mu = 0$ and $n = 1/4$.

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We consider the curvature energy functional

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Euler-Lagrange equation

Regardless of the boundary conditions, any critical curve for $\mathbf{\Theta}$ must satisfy

$$\kappa^{3/4}\frac{d^2}{ds^2}\left(\frac{1}{\kappa^{3/4}}\right) - 3\kappa^2 + \rho = 0\,.$$

We will call them, simply, critical curves.

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Proposition (Langer & Singer, 1984)

Consider $N^2(\rho)$ embedded as a totally geodesic surface of $N^3(\rho)$. Then, the vector fields

$$\begin{aligned} \mathcal{I} &= \frac{1}{4\kappa^{3/4}} B \,, \\ \mathcal{J} &= -\frac{3}{4}\kappa^{1/4} T + \frac{d}{ds}\left(\frac{1}{4\kappa^{3/4}}\right) N \end{aligned}$$

are Killing vector fields along critical curves.

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2. Let's denote by ξ the (unique) extension to a Killing vector field of $N^3(\rho)$. (It can be assumed to be: $\xi = \lambda_1 X_1 + \lambda_2 X_2$.)

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3. Since $N^3(\rho)$ is complete, the one-parameter group of isometries determined by ξ is $\{\phi_t \,,\, t \in \mathbb{R}\}$.
4. We construct the binormal evolution surface **(Garay & —, 2016)**

$$S_\gamma := \{x(s,t) := \phi_t\left(\gamma(s)\right)\} \,.$$

Geometric Properties

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- Since $\gamma(s)$ is a critical curve for $\mathbf{\Theta}$,

Theorem **(Montaldo & —, 2020)**

The binormal evolution surface S_γ is a proper biconservative surface. It verifies:

$$3\kappa_1 + \kappa_2 = 0\,.$$

Closure Conditions

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Closure Conditions

Let $\gamma(s) \subset N^2(\rho)$ be a critical curve for $\mathbf{\Theta}$ with periodic curvature. Then, $\gamma(s)$ is closed if and only if

$$\Lambda(d) = 12 \int_0^{\varrho} \frac{\kappa^{7/4}}{16d\kappa^{3/2} - \rho}\, ds$$

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equals 0 for $\rho \leq 0$, or $2n\pi/(m\sqrt{\rho d})$ for $\rho > 0$.

Existence of Closed Biconservative Surfaces

Proposition (Montaldo & —, 2020)

There are not closed non-CMC biconservative surfaces in $N^3(\rho)$ with $\rho \leq 0$.

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Proposition (Montaldo & —, 2020)

Critical curves for $\mathbf{\Theta}$ in $\mathbb{S}^2(\rho)$ have periodic curvature.

Idea of the Proof

1. Let $x = \kappa^{1/2}$ and $y = x'$, then the first integral of the Euler-Lagrange equation reads

$$y^2 = \frac{4}{9}x^2\left(16dx^3 - 9x^4 - \rho\right) = \frac{4}{9}x^2 Q(x)\,.$$

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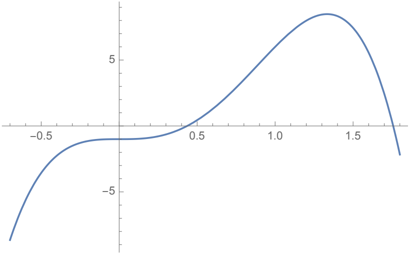

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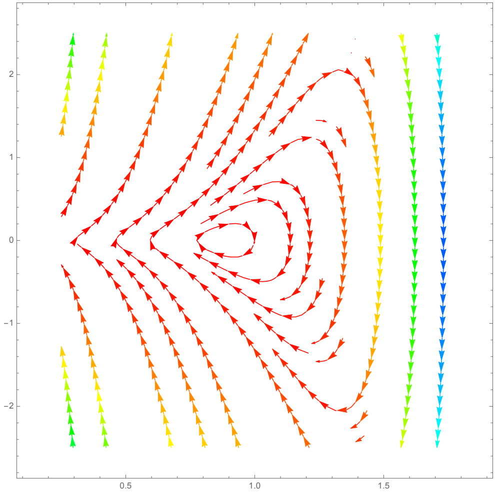

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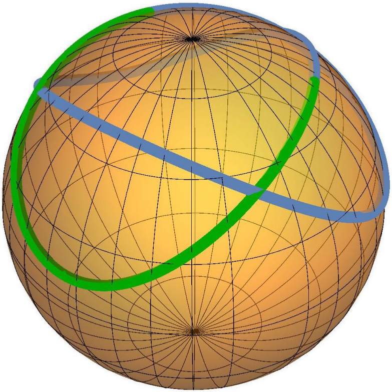
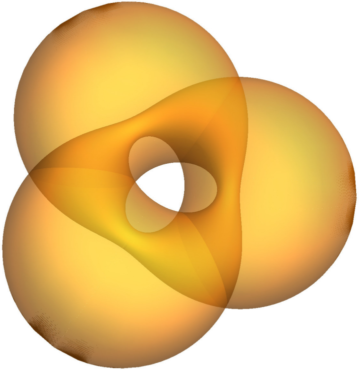
- For any m and n such that, $m < 2n < \sqrt{2}m$, we have a closed non-CMC biconservative surface.

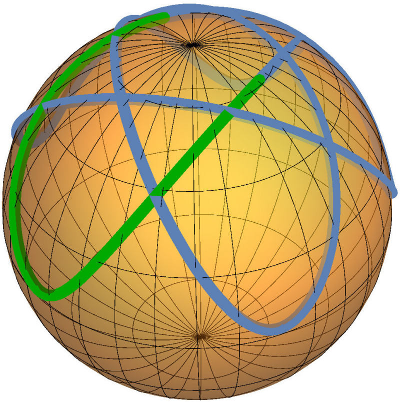
Critical Curve for Θ ($m = 3$ and $n = 2$)



Closed Biconservative Surface ($m = 3$ and $n = 2$)



Critical Curve for $\boldsymbol{\Theta}$ ($m = 5$ and $n = 3$)



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