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Planar p-Elasticae and Rotational Linear Weingarten Surfaces

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Introduction

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  Helfreich-Canham Models in Biophysics, Worldsheets for Kleinert-Polyakov Action in String Theory, Fluid Dynamics..

Index

Index

Index

Index

Index

1. Planar p-Elasticae
2. Binormal Evolution of p-Elasticae
3. Rotational Linear Weingarten Surfaces
4. Remarkable Particular Cases

Planar p-Elasticae

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1. Varational Problem

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2. Involved Classical Energies

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5. First Integral of Euler-Lagrange

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- Take into account that $\kappa = \mu$ would be a global minimum if we were considering $L^1([0, L])$ as the space of curves.

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  And, the critical curves are elastic curves.
- If $p = \frac{1}{2}$ and $\mu = 0$, we have a variational problem studied by Blaschke in 1930, obtaining catenaries.

Euler-Lagrange Equation

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The Euler-Lagrange equation for the curvature energy functional $\mathbf{\Theta}(\gamma) = \int_\gamma (\kappa - \mu)^p$, in $\mathbb{R}^2$ with $p \neq 0, 1$ can be written as

$$\frac{d^2}{ds^2}\left((\kappa-\mu)^{p-1}\right) + \kappa^2 (\kappa-\mu)^{p-1} - \frac{1}{p}\kappa (\kappa-\mu)^p = 0\,.$$

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Generalized EMP Equation [3]

The Euler-Lagrange equation is a generalized EMP equation. Indeed, for $p = \frac{1}{2}$, we get the proper EMP equation

$$\phi'' + \mu^2 \phi = \frac{1}{\phi^3}\,.$$

Killing Fields along p-Elasticae

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Killing Vector Fields along γ [1]

The vector fields along γ defined by

$$\begin{aligned} \mathcal{I} &= (\kappa-\mu)^{p-1}\,B\,, \\ \mathcal{J} &= ((p-1)\kappa+\mu)\,(\kappa-\mu)^{p-1}\,T + p\,\frac{d}{ds}\left((\kappa-\mu)^{p-1}\right)N \end{aligned}$$

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are Killing vector fields along γ, if and only if, γ verifies the Euler-Lagrange equation.

First Integral of Euler-Lagrange

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Theorem [3]

The derivative of the function $\langle \mathcal{J}, \mathcal{J} \rangle$ along the critical curves is zero. Thus, we have that

$$p^2 |\mathcal{J}|^2 = d \,,$$

for any positive constant d.

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Therefore, we can integrate the Euler-Lagrange equation, obtaining

$$(\kappa')^2 = \frac{(\kappa - \mu)^2}{p^2 (p-1)^2} \left(d \, (\kappa - \mu)^{2(1-p)} - ((p-1)\kappa + \mu)^2 \right) \, .$$

Binormal Evolution of p-Elasticae

1. Associated Killing Vector Field

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3. Geometric Properties of this Binormal Evolution Surface

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Thus, any planar p-Elasticae has two associated Killing vector fields, which extend $\mathcal{I}$ and $\mathcal{J}$.

- Killing vector fields in $\mathbb{R}^3$ are the infinitesimal generators of isometries.
- Any Killing vector field in $\mathbb{R}^3$ can be assumed to be of helical type

$$\lambda_1 X + \lambda_2 V \, .$$

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3. Since $\mathbb{R}^3$ is complete, we have the one-parameter group of isometries determined by the flow of ξ is given by $\{\phi_t, t \in \mathbb{R}\}$.

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Take γ any planar p-Elasticae contained in any totally geodesic surface of $\mathbb{R}^3$.

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3. Since $\mathbb{R}^3$ is complete, we have the one-parameter group of isometries determined by the flow of ξ is given by $\{\phi_t, t \in \mathbb{R}\}$.
4. Now, construct the surface $S_\gamma := \{x(s,t) := \phi_t(\gamma(s))\}$.

Geometric Properties of this BES

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Theorem [1]

Let γ be a planar curve, then, the BES with initial condition γ is either, a flat isoparametric surface, if κ is constant; or a rotational surface, if κ is not constant.

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Theorem [4]

Let γ be a planar p-Elasticae, then, the BES generated by γ verifies $\kappa_1 = a\,\kappa_2 + b$, for

$$a = \frac{p}{p-1}\,, \quad b = \frac{-\mu}{p-1}\,.$$

Rotational Linear Weingarten Surfaces

1. Weingarten Surfaces

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2. Classification of Rotational Linear Weingarten Surfaces

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3. Characterization of Profile Curves

Weingarten Surfaces

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- Umbilical Surfaces (Plane and Sphere)
- Isoparametric Surfaces (Circular Cylinders)
- Constant Mean Curvature Surfaces (Rotational Case: Delaunay Surfaces)

Classification $(b = 0)$

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The rotational linear Weingarten surfaces satisfying the relation $\kappa_1 = a\kappa_2$, $a \neq 0$, are planes, ovaloids and catenoid-type surfaces.

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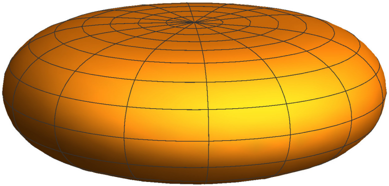
Moreover,

- Case $a > 0$. The rotational surface is an ovaloid.



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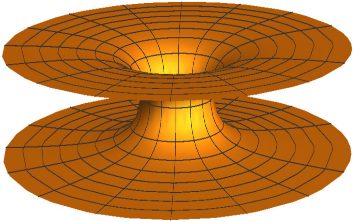
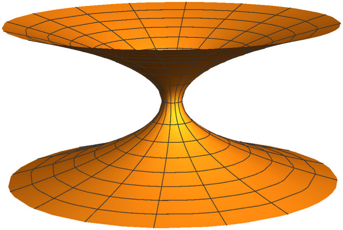


(A) $a < -1$



(B) $a \in [-1, 0)$

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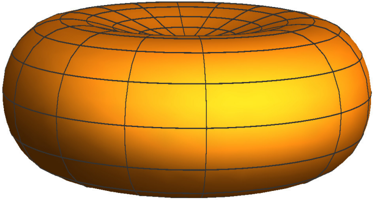
- Vesicle-Type Surfaces



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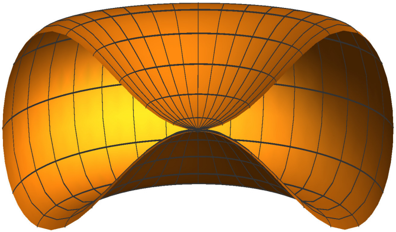

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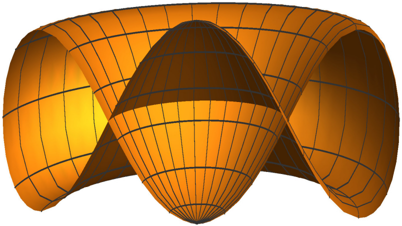
- Immersed Spheroid



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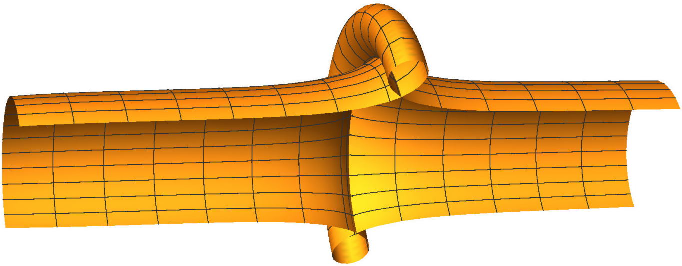

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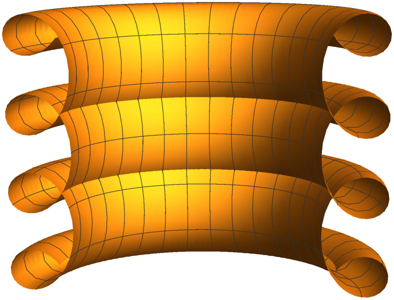

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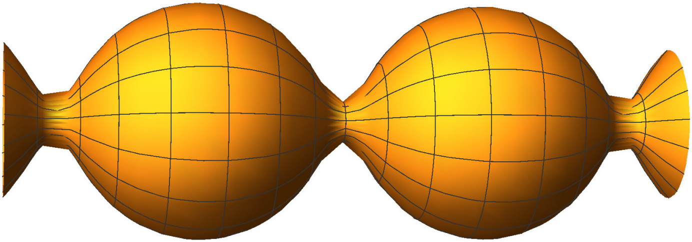
Let $a < 0$ and $b \neq 0$. The rotational linear Weingarten surfaces are unduloid-type, circular cylinders, spheres and nodoid-type.

- Unduloid-Type Surfaces



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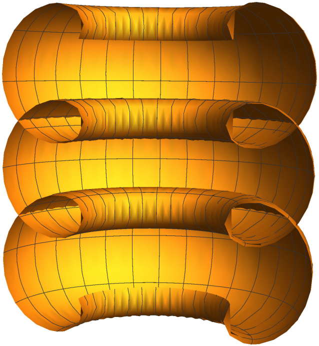

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Theorem [4]

Let M be a rotational linear Weingarten surface and let $\gamma(s)$ be its profile curve. Then, if $a \neq 1$, γ is a planar p-Elastic curve for

$$\mu = \frac{-b}{a-1}\,, \quad p = \frac{a}{a-1}\,.$$

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Every rotational linear Weingarten surface (with $a \neq 1$) admits a geodesic foliation by planar p-Elasticae.

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Every rotational linear Weingarten surface (with $a \neq 1$) admits a geodesic foliation by planar p-Elasticae.

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The evolution under the binormal flow of any planar p-Elasticae generates rotational linear Weingarten surfaces.

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Solving the Euler-Lagrange equations, we obtain that the non-geodesic planar elastic curves have curvature given by

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- $\kappa_o = \kappa_o(d)$ is a constant (the maximum curvature) and cn denotes the Jacobi cosine.

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- Thus, after rotating we obtain the parametrization of Mylar Balloons:

$$x(s,\theta) = \frac{1}{\sqrt{d}} \left(2\kappa \cos\theta,\, 2\kappa \sin\theta\,, \int \kappa^2\, ds \right) \,,$$

where $\kappa(s)$ is the curvature of γ.

Extended Blaschke's Energy

Take $p = \frac{1}{2}$ in the p-Elastic energy, that is,

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$$\kappa(s) \quad = \quad \frac{2\mu(\omega^2 + \omega \sin 2\mu s)}{1 + \omega^2 + 2\omega \sin 2\mu s}\,,$$

where $\omega^2 = 1 + \frac{\mu}{d}$.

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A Delaunay surface is, precisely, a binormal evolution surface with a critical curve for the extended Blaschke's energy as initial condition.

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Characterization of Delaunay Surfaces [2]

A Delaunay surface is, precisely, a binormal evolution surface with a critical curve for the extended Blaschke's energy as initial condition. Moreover, the constant mean curvature is given by

$$H = -\mu.$$

References

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THE END

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