Existence and Properties of Closed Free p-Elastic Curves

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- **1744**: L. Euler described the shape of planar elastic curves (partially solved by Jacob Bernoulli 1692–1694).

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- Case $p > 2$. (Applications: Willmore-Chen submanifolds, string theories,...)

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- Cases $p = (n-2)/(n+1)$. Arise in the theory of biconservative hypersurfaces. (Montaldo & P., 2020; Montaldo, Oniciuc & P., 2022).

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The study of free p-elastic curves is a central topic in Differential Geometry and Calculus of Variations.

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- For a general value of $p \in \mathbb{R}$, (non-trivial) closed free p-elastic curves in $\mathbb{S}^2$ may not exist. Indeed, when $p > 2$ is natural, the only closed free p-elastic curves are geodesics. (Arroyo, Garay & Mencía, 2003).

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- For a general value of $p \in \mathbb{R}$, (non-trivial) closed free p-elastic curves in $\mathbb{S}^2$ may not exist. Indeed, when $p > 2$ is natural, the only closed free p-elastic curves are geodesics. (Arroyo, Garay & Mencía, 2003).
- Finding (non-trivial) closed free p-elastic curves is interesting but not trivial.

Variational Problem

Let $p \in \mathbb{R}$ and consider the functionals

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acting on the space of smooth immersed spherical curves. When $p \in \mathbb{R} \setminus \mathbb{N}$, we restrict to the subspace of convex curves.

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The Euler-Lagrange Equation

A critical point γ of $\mathbf{\Theta}_p$ must satisfy

$$p \frac{d^2}{ds^2}\left(\kappa^{p-1}\right) + (p-1)\kappa^{p+1} + p\kappa^{p-1} = 0 \, .$$

Critical Circles and First Integral

Critical Circles

If γ is a critical point of $\boldsymbol{\Theta}_p$ with constant curvature κ then, either $p \in \mathbb{N}$ $(p \neq 1)$ and γ is a geodesic, or $p \in [0,1)$ and

$$\kappa = \sqrt{\frac{p}{1-p}}\,.$$

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First Integral

If γ is a critical point of $\mathbf{\Theta}_p$ with non-constant curvature κ then

$$p^2(p-1)^2\kappa^{2(p-2)}\left(\kappa'\right)^2 + (p-1)^2\kappa^{2p} + p^2\kappa^{2(p-1)} = a \in \mathbb{R}^+ \,,$$

must hold. (The case $p = 2$ is special.)

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- Consequently, a p-elastic curve ($p \neq 2$) has periodic curvature if and only if $p \in (0, 1)$.

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4. It is closed if and only if the angular progression is a rational multiple of 2π.

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5. **Theorem**. Any arch is unstable. (Gruber, P. & Toda, submitted).

Closure Condition

Let γ be a p-elastic curve ($p \neq 2$) with periodic curvature. Then γ is closed if and only if

$$\Lambda_p(a) := (1-p)\sqrt{a} \int_0^{\varrho} \frac{\kappa^{2-p}}{a\,\kappa^{2(1-p)} - p^2}\, ds = 2\pi q\,,$$

where $q \in \mathbb{Q}$ and $a \in (a_*, \infty)$.

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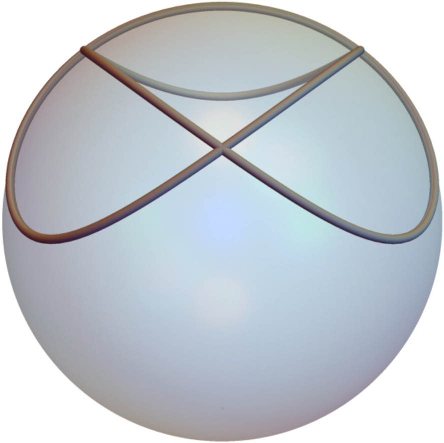
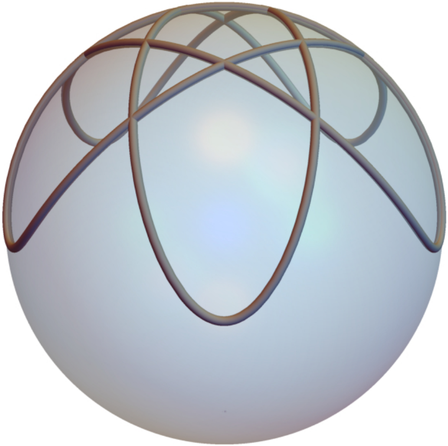
Theorem (Gruber, P. & Toda, submitted)

Let n and m be two relatively prime natural numbers satisfying $m < 2n < \sqrt{2}\, m$. Then, for every $p \in (0,1)$, there exists a closed p-elastic curve with non-constant curvature.

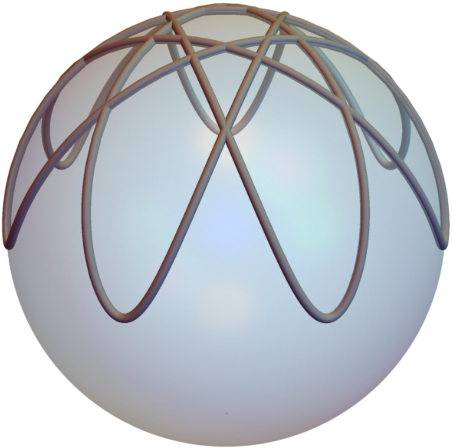
Example of Type $(2, 3)$



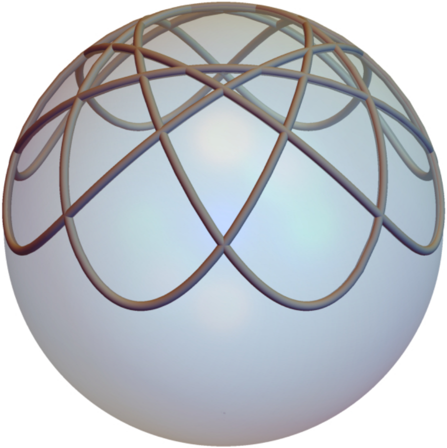
Example of Type $(3, 5)$



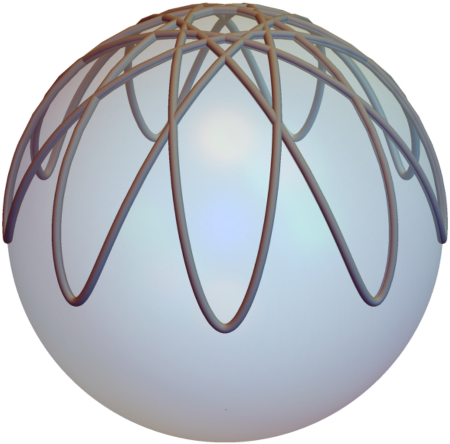
Example of Type $(4, 7)$



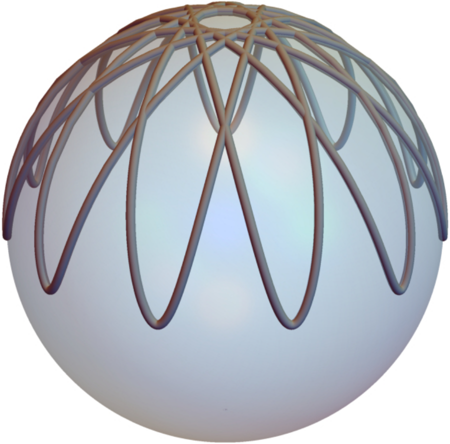
Example of Type $(5, 8)$



Example of Type $(5, 9)$



Example of Type $(6, 11)$



Evolution on the Energy Parameter

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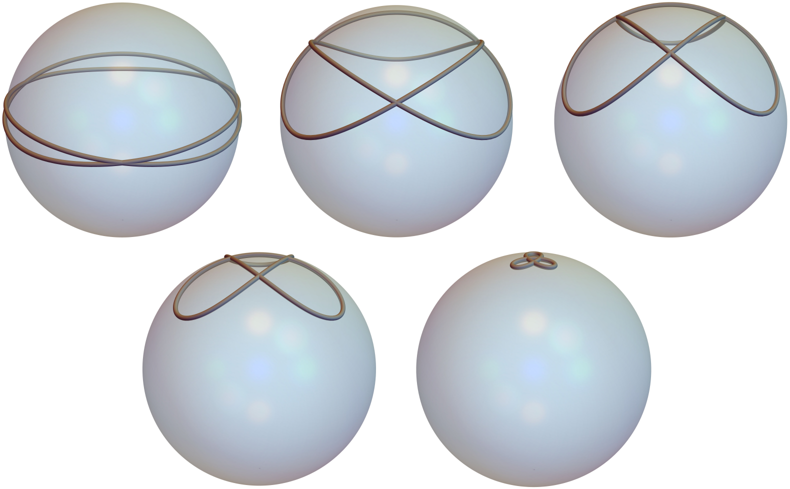

THE END

Thank You!