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Invariant Surfaces in $\mathbb{S}^3$ Based on Generalized Elastic Curves

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- **<u>1744</u>: L. Euler.**
  Described the shape of planar elasticae (with constraint on the length).
  Partially solved by J. Bernoulli, 1692-1694.

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1985: U. Pinkall

Link between Willmore surfaces and elastica.

Scheme

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1. Part I. Generalized Elastic Curves

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3. Part III. Vertical Lifts

Part I

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For fixed real constants μ, $p \in \mathbb{R}$, we consider the biparametric family of curvature energy functionals

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- We are mainly interested on the space of closed curves.

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- If $p = 2$ and $\mu \neq 0$, $\mathbf{\Theta}$ is the bending energy (circular at rest).
- If $p = 1/2$, we obtain an extension of an energy studied by Blaschke in 1930.

Variational Problem

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For simplicity we denote $P(\kappa) := (\kappa - \mu)^p$. Then,

Euler-Lagrange Equation

Regardless of the boundary conditions, a critical curve γ in $\mathbb{S}^2(\rho)$ satisfies

$$\dot{P}_{ss} + \dot{P}\left(\kappa^2 + \rho\right) - \kappa P = 0 \, . \qquad \left(\dot{P} \equiv \frac{dP}{d\kappa}\right)$$

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Vector Fields Along Critical Curves

Consider $\mathbb{S}^2(\rho)$ embedded as a totally geodesic surface of $\mathbb{S}^3(\rho)$. Then, we have

$$\mathcal{J} = \left(\kappa \dot{P} - P\right) T + \dot{P}_s N\,, \qquad \mathcal{I} = \dot{P} B$$

where $\{T, N, B\}$ denotes the Frenet frame of γ in $\mathbb{S}^3(\rho)$.

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3. In this case, using spherical coordinates in $\mathbb{S}^2(\rho) \subset \mathbb{R}^3$, we get the following parametrization of the critical curves:

$$\gamma(s) = \frac{1}{\sqrt{\rho\, d}} \left(\sqrt{\rho}\, \dot{P}, \sqrt{d - \rho\, \dot{P}^2}\, \sin \Psi(s), \sqrt{d - \rho\, \dot{P}^2}\, \cos \Psi(s) \right)$$

where

$$\Psi(s) = \sqrt{\rho\, d} \int \frac{\kappa \dot{P} - P}{d - \rho\, \dot{P}^2}\, ds \,.$$

Closure Condition

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- Assume $\kappa(s)$ is periodic (of period ϱ). Then,

Closure Condition

The critical curve $\gamma(s)$ in $\mathbb{S}^2(\rho)$ is closed, if and only if,

$$\Lambda(d) = \sqrt{\rho\, d} \int_o^{\varrho} \frac{\kappa \dot{P} - P}{d - \rho\, \dot{P}^2}\, ds = 2\frac{n}{m}\pi\,,$$

for any integers n and m.

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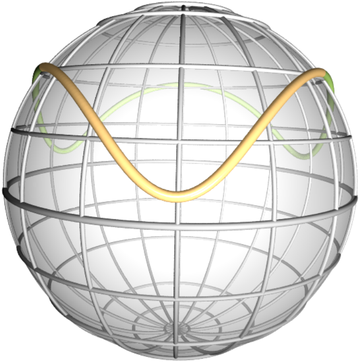
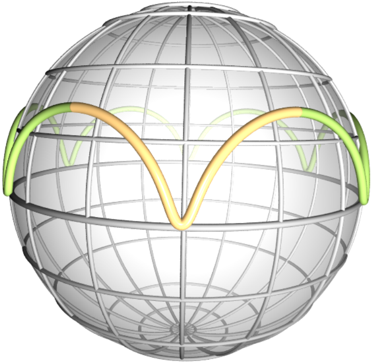
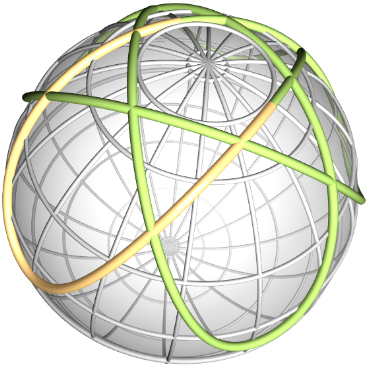


(C) $\mu = -1$



(D) $\mu = -2$

Geometric Description (2)



(E) $\mu = -0.1$

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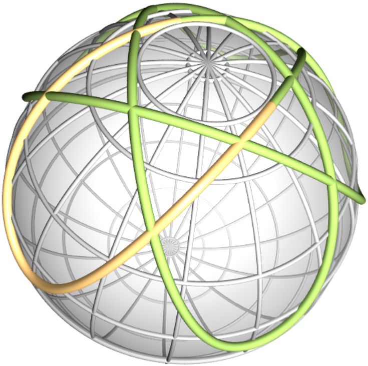

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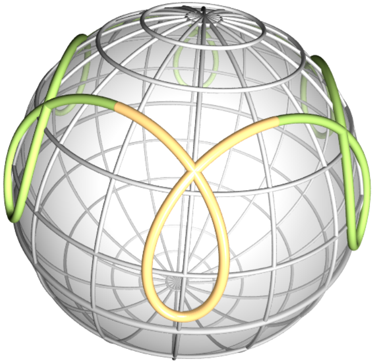
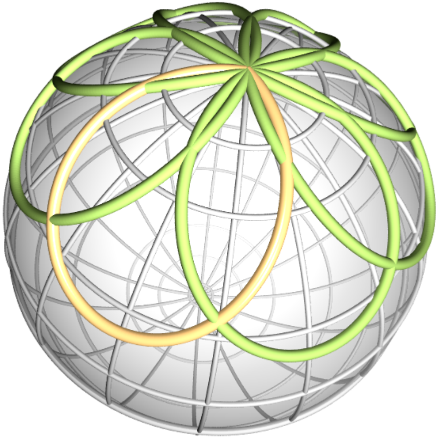


(H) $\mu = 1$

Geometric Description (3)



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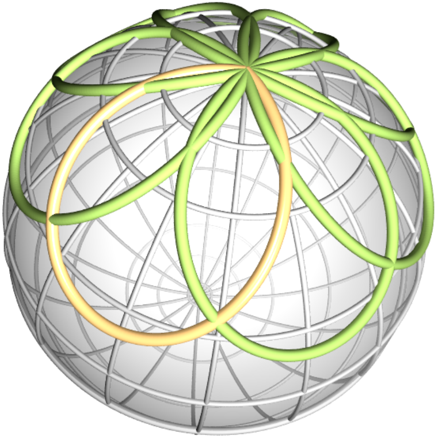

- They never cut the axis $x_1 = 0$ (the equator), since $\dot{P} = \frac{1}{2\sqrt{\kappa - \mu}} > 0$.

Part II

Binormal Evolution

Killing Vector Fields

A vector field W along γ, which infinitesimally preserves unit speed parametrization is said to be a Killing vector field along γ if it evolves in the direction of W without changing shape, only position. That is, the following equations hold

$$W(v) = W(\kappa) = 0$$

along γ. **(Langer & Singer, 1984)**

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The vector fields $\mathcal{I}$ and $\mathcal{J}$ are Killing vector fields along critical curves.
(We are mainly interested in $\mathcal{I}$.)

- Killing vector fields along γ can be extended to Killing vector fields on the whole $\mathbb{S}^3(\rho)$. The extension is unique.

Binormal Evolution Surfaces

Let $\gamma(s) \subset \mathbb{S}^2(\rho)$ be any generalized elastic curve. (We consider $\mathbb{S}^2(\rho) \subset \mathbb{S}^3(\rho)$ and γ being planar, i.e. $\tau = 0$.)

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4. We construct the binormal evolution surface **(Garay & —, 2016)**

$$S_\gamma := \{x(s,t) := \phi_t(\gamma(s))\} \,.$$

Geometric Properties

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Theorem (Arroyo, Garay & —, 2017)

The binormal evolution surface S_γ is either a flat isoparametric surface (when $\kappa(s) = \kappa_o$ is constant); or, it is a rotational surface (when $\kappa(s)$ is not constant).

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Theorem (—, 2018)

The binormal evolution surface S_γ is a linear Weingarten surface. It verifies:

$$\kappa_1 = a\kappa_2 + b\,, \qquad (\kappa_i \text{ principal curvatures})$$

for $a = p/(p-1)$ and $b = -\mu/(p-1)$.

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- Constant Mean Curvature Surfaces
  (Rotational: Delaunay Surfaces in $\mathbb{S}^3(\rho)$)

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The profile curve γ of a rotational linear Weingarten surface of $\mathbb{S}^3(\rho)$ (for $a \neq 1$) is a planar ($\gamma(s) \subset \mathbb{S}^2(\rho)$) generalized elastic curve for $\mu = -b/(a-1)$ and $p = a/(a-1)$.

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- The case $a = 1$; rotational surfaces with constant skew curvature. **(López & —, 2020)**

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Specializing previous characterization we get

Theorem (Arroyo, Garay & —, 2018)

A rotational surface with constant mean curvature H of $\mathbb{S}^3(\rho)$ is, locally, a binormal evolution surface with initial condition a generalized elastic curve in $\mathbb{S}^2(\rho)$ for $p = 1/2$ and $\mu = -H$, i.e. for the extended Blaschke's energy

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Basically, we need to understand these critical curves:

- If $\kappa = \mu$, we have global minima (acting on the space L^1).

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3. A Hopf torus

$$\mathbb{S}^1\left(\sqrt{\rho + \kappa_o^2}\right) \times \mathbb{S}^1\left(\frac{\sqrt{\rho}}{\kappa_o}\sqrt{\rho + \kappa_o^2}\right)$$

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   if $\kappa(s) = \kappa_o = -|H| + \sqrt{\rho + H^2}$.

4. A binormal evolution surface shaped on γ (planar critical curve of extended Blaschke's energy for $|\mu| = |H|$ with non-constant curvature).

Illustration (1)

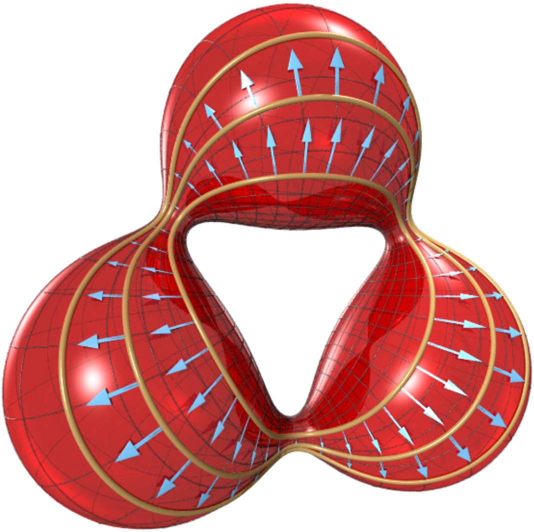
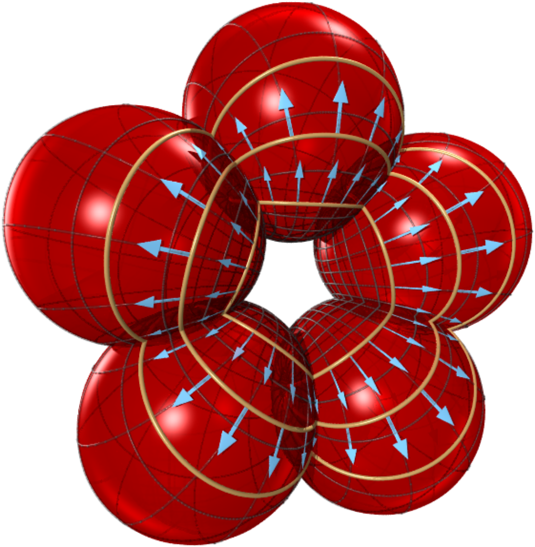


Illustration (2)



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- For this last theorem, we need to consider $n = 1$ in the closure condition of critical curves, which yields to an already known condition.

Embedded CMC Tori in $\mathbb{S}^3(\rho)$

Theorem (Perdomo, 2010)

For any $m > 1$ and any H such that

$$|H| \in \left(\sqrt{\rho} \cot \frac{\pi}{m}, \sqrt{\rho} \frac{m^2 - 2}{2\sqrt{m^2 - 1}} \right)$$

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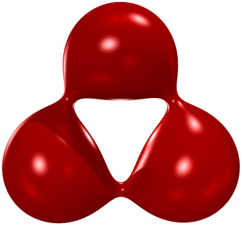

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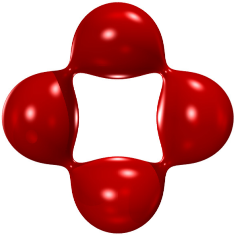

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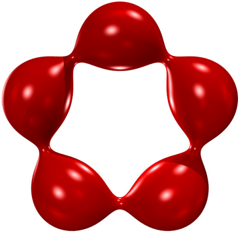

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Lawson's Conjecture (Lawson, 1970)

The only embedded minimal tori in $\mathbb{S}^3(\rho)$ is the Clifford torus. (Recently proved in **[Brendle, 2013]**.)

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A surface $S \subset \mathbb{S}^3(\rho)$ is said to be biconservative if it satisfies

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$$3\kappa_1 + \kappa_2 = 0\,.$$

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Theorem (Montaldo & —, *submitted*)

All non-CMC biconservative surfaces of $\mathbb{S}^3(\rho)$ are, locally, binormal evolution surfaces with the initial condition critical for

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Part III

Vertical Lifts

Hopf Tori

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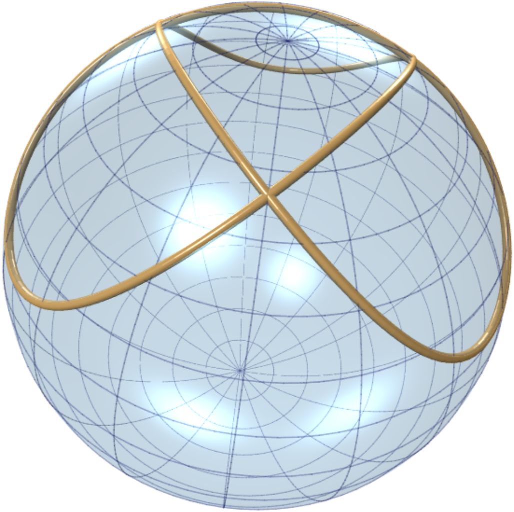
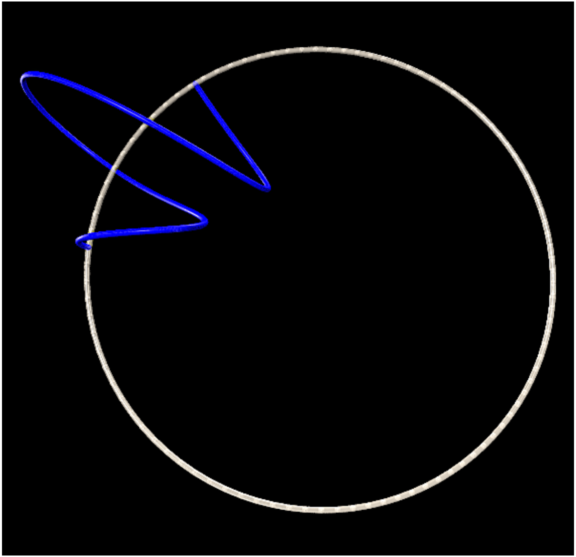
   (A condition on the enclosed area is essential, **(Arroyo, Barros & Garay, 2000)**.)

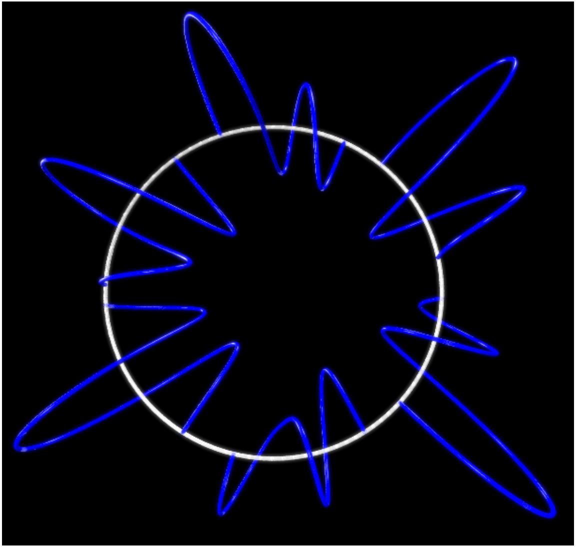
Horizontal Lift (Base Curve)



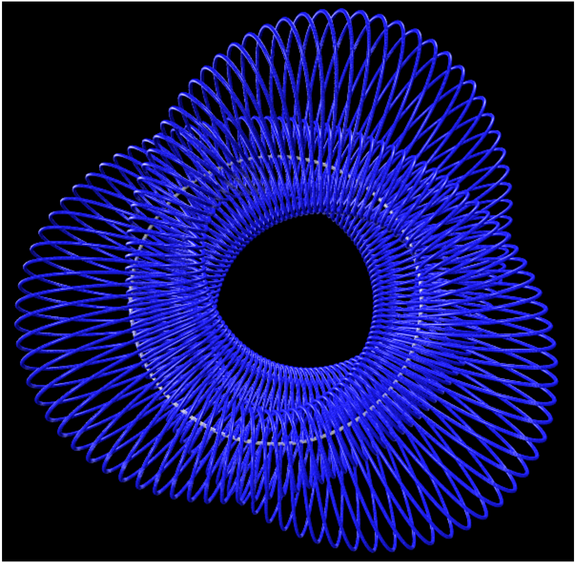
Horizontal Lift (One Lift)



Horizontal Lift (Six Lifts)



Horizontal Lift (One Hundred Lifts)



Surface Energies

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Consider the biparametric family of energies

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- These energies are kind of p-Willmore energies, **(Gruber, Toda & Tran, 2019)**.
- We introduce the notation $P(H) := (H-\lambda)^p$.

Euler-Lagrange Equation

A critical point of $\mathcal{F}(S)$ in $\mathbb{S}^3(\rho)$ satisfies

$$\Delta P' + 2P'\left(2H^2 - K + 2\rho\right) - 4PH = 0$$

where $P' \equiv dP/dH$.

Critical p-Willmore Hopf Tori

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For a Hopf tori $\widetilde{S}_\gamma$, the mean curvature is given by

$$H = \frac{1}{2}\left(\kappa \circ \widetilde{\pi}\right) \qquad \left(\widetilde{\pi} : \mathbb{S}^3(\rho) \to \mathbb{S}^2(4\rho)\right)$$

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- Let γ be a closed curve in $\mathbb{S}^2(4\rho)$.
- Using H and the Symmetric Criticality Principle of Palais **(Palais, 1979)**, we get

Theorem (—, *submitted*)

The Hopf torus $\widetilde{S}_\gamma = \widetilde{\pi}^{-1}(\gamma)$ based on γ is a critical point of $\mathcal{F}(S)$ if and only if γ is a generalized elastic curve in $\mathbb{S}^2(4\rho)$ with $\mu = \lambda/2$, i.e.,

$$\mathbf{\Theta}(\gamma) = \int_\gamma (\kappa - \mu)^p \,.$$

Correspondence Result

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- There exists a correspondence between (rotational) linear Weingarten tori and critical p-Willmore Hopf tori in $\mathbb{S}^3$.

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We recover the Blaschke's curvature energy ($p = 1/2$ and $\mu = 0$):

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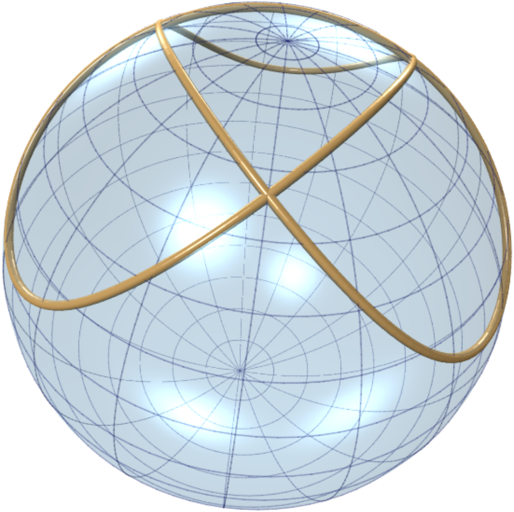
5. We consider this critical curve, $\gamma_{2,3}$.

Critical Curve $\gamma_{2,3}$



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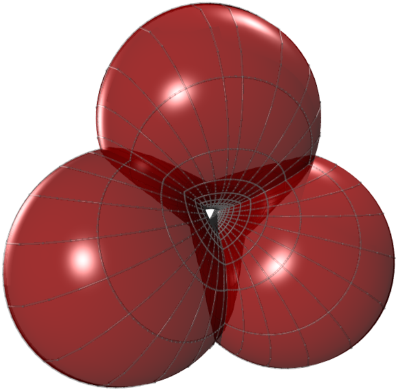

(o) Minimal (Rotational) Torus

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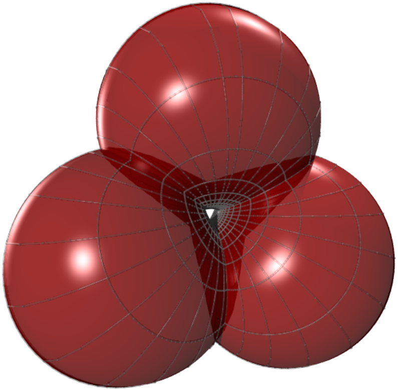

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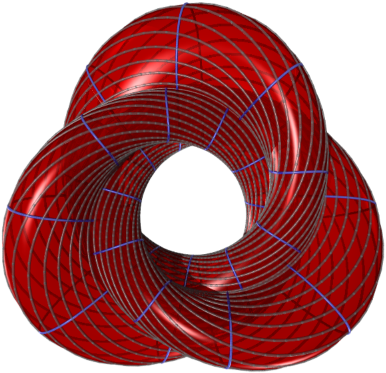

(R) Hopf Torus

Consequences

Corollary

The Hopf torus $\widetilde{S}_\gamma$ based on $\gamma_{2,3}$ is critical for

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THE END

Thank You!

Acknowledgements: Research partially supported by MINECO-FEDER, PGC2018-098409-B-100 and by Gobierno Vasco, IT1094-16.