LECTURE NOTES

Math 3342, Mathematical Statistics

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1 Descriptive Statistics (Chapter 1)

1.1 Measures of Location

Definition 1.1 *The arithmetic mean $\bar{x}$ of a sample of observations $x_1,...,x_n$ is given by*

$$\bar{x} = \frac{x_1 + ... + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i \,.$$

Definition 1.2 *The geometric mean $\bar{x}_g$ of a sample of observations $x_1,...,x_n$ is defined by*

$$\bar{x}_g = (x_1 \cdots x_n)^{1/n} = \left(\prod_{i=1}^{n} x_i\right)^{1/n} .$$

Definition 1.3 *The harmonic mean $\bar{x}_h$ of a sample of observations $x_1,...,x_n$ is*

$$\bar{x}_h = \frac{n}{\frac{1}{x_1} + ... + \frac{1}{x_n}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{-1}\right)^{-1} .$$

Proposition 1.4 *For any sample of observations the following inequalities hold:*

$$\bar{x} \geq \bar{x}_g \geq \bar{x}_h \,.$$

Example 1.5 *In a course with four exams your grades are 4, 7, 3, 6 (out of ten). If all of them are worth the same, compute the means and decide which one you prefer. (Answer: $\bar{x} = 5$, $\bar{x}_g = 4.74$ and $\bar{x}_h = 4.48$.)*

Remark 1.6 *There are other possibilities to define means. Throughout this course we will focus on the (arithmetic) mean. When the mean is computed over all the population is said the (population) mean and denoted by μ.*

Definition 1.7 *A median of a sample is the value such that at most half of the sample is less than it and at most half is greater than it.*

Remark 1.8 *To compute the median, we first order the n observations $x_1,\ldots,x_n$ from smallest to largest. Then, the median $\widetilde{x}$ is the single middle value if n is odd, or the average of the two middle values if n is even.*

Definition 1.9 *The mode is the value that appears most often in a sample.*

Example 1.10 *The salaries of four people per month are:* \$500,000*,* \$200*,* \$300 *and* \$100*. Compute the mean and the median. Is it accurate to say that people is rich? (Answer: $\bar{x} = 125,100$ and $\widetilde{x} = 250$.)*

Remark 1.11 *A trimmed mean can be used to avoid this problems, eliminating the smallest and largest values of the sample.*

Remark 1.12 *There are other measures of location: for instance, quartiles and percentiles. Roughly speaking, quartiles divide the data set into four equal parts, while percentiles divide it in 100 parts.*

1.2 Measures of Variability

Definition 1.13 *The variance of a sample of observations $x_1,\ldots,x_n$ is given by*

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 ,$$

where $\bar{x}$ is the (arithmetic) mean.

Definition 1.14 *The variance of a population $x_1,\ldots,x_n$ is given by*

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2 ,$$

where μ is the (arithmetic) mean of the population.

Remark 1.15 *The difference in the denominator is just a corrector factor which will simplify the formulas to make inference.*

Definition 1.16 *The standard deviation of a sample of observations $x_1,\ldots,x_n$ is the (positive) square root of the variance, i.e.,*

$$s = \sqrt{s^2} .$$

Definition 1.17 *The* <u>*standard deviation*</u> *of a population is the (positive) square root of the variance of the population, i.e.,*

$$\sigma = \sqrt{\sigma^2}\,.$$

Proposition 1.18 *Given an arbitrary sample of observations* $x_1,\ldots,x_n$ *the following relation holds:*

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2 .$$

2 Probability (Chapter 2)

2.1 Sample Spaces and Events

Definition 2.1 *An experiment is any process whose outcome is subject to uncertainty.*

Definition 2.2 *The sample space of an experiment, denoted by $\mathcal{S}$, is the set of all possible outcomes of that experiment.*

Example 2.3 *Determine the sample space of the experiment consisting on tossing three coins. (Heads and Tails). (Answer: $\mathcal{S} = \{HHH, HHT, HTT, HTH, THH, THT, TTT, TTH\}$.)*

Definition 2.4 *An event is any collection (subset) of outcomes contained in the sample space $\mathcal{S}$. An event is simple if it consists of exactly one outcome and compound if it consists of more than one outcomes.*

Example 2.5 *Imagine three vehicles taking an exit from a highway. They may turn left (L) or right (R). The sample space is*

$$\mathcal{S} = \{LLL, LLR, LRR, LRL, RLL, RLR, RRR, RRL\}.$$

A simple event could be $\{LRR\}$, while examples of compound events are:

1. *Exactly one vehicle turns right:*

$$\{LLR, LRL, RLL\}.$$

2. *At least one vehicle turns left:*

$$\{LLL, LLR, LRR, LRL, RLL, RLR, RRL\}.$$

3. *The three vehicles turn in the same direction:*

$$\{LLL, RRR\}.$$

Remark 2.6 *Roughly speaking, the probability of an event to occur is the number of possible outcomes in the event divided by the number of total outcomes.*

Definition 2.7 *The complement of an event A, denoted A', is the set of all outcomes in $\mathcal{S}$ that are not contained in A.*

Definition 2.8 *The union of two events A and B, $A \cup B$, is the event consisting of all outcomes that are either in A, or in B, or in both.*

Definition 2.9 *The intersection of two events A and B, $A \cap B$, is the event consisting of all outcomes that are in both A and B.*

Definition 2.10 *If two events have empty intersection, i.e., $A \cap B = \emptyset$, they are said to be mutually exclusive or disjoint.*

Remark 2.11 *A useful representation of events and manipulations can be represented using Venn diagrams.*

2.2 Counting Techniques

Definition 2.12 *Consider a set of n distinct elements. An ordered subset is called a permutation. An unordered subset is called a combination.*

Remark 2.13 *A permutation is an ordered combination.*

Example 2.14 *Among 10 athletes, the possible ways to distribute the medals in the Olympic games is a permutation of size 3 among 10 individuals, $P_{3,10}$.*

Example 2.15 *Among 10 flavors of ice cream, the possible variations we have to get three scoops of different flavors is a combination of size 3 among 10 objects, $C_{3,10}$.*

Remark 2.16 *There exist both permutations and combinations with repetition, denoted by $P^r_{k,n}$ and $C^r_{k,n}$, respectively.*

Proposition 2.17 (Permutations) *The number of permutations of size k that can be formed from n elements is:*

$$P^r_{k,n} = n^k \,,$$

if repetition is allowed; and,

$$P_{k,n} = \frac{n!}{(n-k)!} \,,$$

if repetition is not allowed.

Remark 2.18 *The symbol ! represents the factorial which is defined by*

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1 = \prod_{i=1}^{n} i \,.$$

By definition, $0! = 1$.

Example 2.19 *How many telephone numbers of 10 digits may exist? (Answer: 10^{10}.)*

Example 2.20 *How many ways are there to distribute the medals (Gold, Silver and Bronze) in the Olympic games among* 10 *athletes? (Answer:* 720.*)*

Proposition 2.21 (Combinations) *The number of combinations of size k that can be formed from n elements is:*

$$C_{k,n} = \binom{n}{k} = \frac{n!}{k!(n-k)!}\,,$$

if repetition is not allowed; and

$$C^r_{k,n} = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}\,,$$

if repetition is allowed.

Remark 2.22 *The number*

$$\binom{n}{k}$$

is called a <u>*combinatorial number*</u>*.*

2.3 Definition and Properties of Probability

Remark 2.23 *The objective of probability is to assign to each event in an experiment a number which gives a precise measure of the chance that the corresponding event will occur.*

Definition 2.24 *Consider an experiment with sample space $\mathcal{S}$. A* <u>*probability measure*</u> *is a function $P : \mathcal{S} \to [0,1]$ such that:*

(i) $P(\mathcal{S}) = 1$*, and*

(ii) If $\{A_i\}_{i=1}^{\infty}$ is an infinite collection of disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)\,.$$

Proposition 2.25 *The probability of the null event is zero, i.e.,* $P(\emptyset) = 0$.

Remark 2.26 *Due to this proposition, the second Axiom in the definition is also valid for finite collection of disjoint events.*

Proposition 2.27 *For any event A, the probability of the complement event is*

$$P(A') = 1 - P(A)\,.$$

Example 2.28 (Birthday Problem) *The birthday problem asks for the probability that, among n randomly selected people, at least two share a birthday (we do not consider February 29).*

1. *How many people are needed for that the probability exceeds* 0.5*? (Answer: 23.)*
2. *What is the probability that, among* 31 *people, at least two share a birthday? (Answer: 0.73.)*

Proposition 2.29 *For any two events A and B, the probability of the union is*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)\,.$$

Example 2.30 (Monty Hall Problem (Part I)) *Suppose you are on the game show "Let's Make a Deal" whose host was Monty Hall. You are given the choice of three doors. Behind one door is a car, while behind the others, there are goats. Once you pick up a door, the host (who knows what is behind every door) opens another door behind which there is a goat. The host then gives you the opportunity of changing doors.*

1. *What is the probability of winning the car if you stay with your original pick? (Answer: 1/3.)*
2. *What is the probability of winning the car if you switch to the other closed door? (Answer: 2/3.)*

(Hint: Describe the sample space and count all the winning outcomes.)

2.4 Conditional Probability

Definition 2.31 *For any two events A and B with $P(B) > 0$, the conditional probability of A given that B has (already) occurred is defined by*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}\,.$$

Example 2.32 *Suppose that of all individuals buying a certain digital camera,* 60% *include an optional memory card in their purchase,* 40% *include an extra battery, and* 30% *include both a card and a battery. Consider randomly selecting a buyer.*

1. *Given that the selected individual purchased an extra battery, compute the probability that an optional card was also purchased. (Answer:* 0.75*.)*
2. *Given that the selected individual purchased a memory card, compute the probability that an extra battery was also purchased. (Answer:* 0.5*.)*

Example 2.33 *Reading habits of a randomly selected reader with respect to "Art" (A), "Books" (B) and "Cinema" (C) are:* $P(A) = 0.14$, $P(B) = 0.23$, $P(C) = 0.37$, $P(A \cap B) = 0.08$, $P(A \cap C) = 0.09$, $P(B \cap C) = 0.13$ *and* $P(A \cap B \cap C) = 0.05$.

1. *Compute the probability of reading about art, given that the reader reads about books. (Answer:* 0.348*).*

2. *Compute the probability that the selected individual reads about art given that he reads at least one of the other two topics. (Answer:* 0.255*).*

3. *Compute the probability of reading about art, given that the selected individual reads at least one topic. (Answer:* 0.286*).*

4. *Compute the probability that the selected individual reads at least one of the first two columns given that he reads about cinema. (Answer:* 0.459*).*

Definition 2.34 *A collection of events* $\{A_i\}_{i=1}^k$ *is said to be* <u>*exhaustive*</u> *if, at least, one of them must occur, i.e., if* $\bigcup_{i=1}^k A_i = \mathcal{S}$.

Remark 2.35 *A collection of mutually exclusive and exhaustive events is a partition of the sample space.*

Theorem 2.36 (The Law of Total Probability) *Let* $\{A_i\}_{i=1}^k$ *be a collection of mutually exclusive and exhaustive events. Then, for any other event* B,

$$P(B) = \sum_{i=1}^{k} P(B|A_i)P(A_i)\,.$$

Theorem 2.37 (Bayes' Theorem) *Let* $\{A_i\}_{i=1}^k$ *be a collection of mutually exclusive and exhaustive events. Then, for any other event* B *with* $P(B) > 0$, *the (posterior) probability of* A_j, *for some* $j = 1, ..., k$, *given that* B *has already occurred is*

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^k P(B|A_i)P(A_i)}\,.$$

Example 2.38 (Monty Hall Problem (Part II)) *Use Bayes' Theorem to solve the Monty Hall problem. (Hint: For simplicity, assume we pick door number* 1 *and Monty opens door number* 2. *The numbers of the doors do not matter, if you prefer think that the doors are called* α, β *and* γ, *and we numbered them as they are appearing. Let* A_i*="The car is behind the door number* i*" and* B*="Monty opens door number* 2*". Compute* $P(A_1|B)$ *and* $P(A_1'|B)$*.)*

Example 2.39 *Only* 1 *in* 100 *adults is afflicted with a virus disease for which a diagnostic test has been developed. The test is such that when an individual actually has the disease, a positive result will occur* 90% *of the time (*<u>*sensitivity*</u>*), whereas an individual without the disease will show a positive test result only* 1% *of the time (*<u>*specificity*</u> *of* 99%*).*

1. *If a randomly selected individual is tested and the result is positive, what is the probability that the individual has the disease? (Answer:* 0.48*).*

2. *If a randomly selected individual is tested and the result is negative, what is the probability that the individual has the disease? (Answer:* 0.001*).*

Example 2.40 *Assume* 8 *in* 10 *adults are afflicted with a virus disease for which a diagnostic test has been developed. The test is such that when an individual actually has the disease, a positive result will occur* 75% *of the time (sensitivity), whereas an individual without the disease will show a positive test result only* 5% *of the time (specificity of* 95%*).*

1. *If a randomly selected individual is tested and the result is positive, what is the probability that the individual has the disease? (Answer:* 0.98*).*

2. *If a randomly selected individual is tested and the result is negative, what is the probability that the individual has the disease? (Answer:* 0.51*).*

Definition 2.41 *Two events A and B are independent if $P(A|B) = P(A)$ and are dependent otherwise.*

Proposition 2.42 *Two events A and B are independent if and only if*

$$P(A \cap B) = P(A)P(B)\,.$$

2.5 Exercises

1. Compute the formula for the probability of the union of three arbitrary events A, B and C.

2. Three people are participating in a competition of art. This competition gives a prize of $200 to the best artist and $100 to the second best. How many options are there to distribute the prizes? (Answer: 6.)

3. Three people are participating in a competition of art. Each participant presents 2 pieces of art. The competition gives a prize of $200 to the best piece of art and $100 to the second best. How many options are there to distribute the prizes? (Answer: 30.)

4. To prepare a salad we have the following ingredients: tomato, carrot, potato and broccoli. How many options are there to prepare the salad with only 2 ingredients? And with 3? And with 1? And with 4? (Answer: 6, 4, 4, 1.)

5. There are 10 people in a chess competition. How many games must be fixed so that each participant plays against all the others exactly once? (Answer: 45.)

6. In a group of 10 students, 6 are men and 4 are women. How many ways are there to choose a committee of 3 people, where at least one is a woman? (Answer: 100.)

7. In a store there are 6 different types of cookies. In how many ways can we choose 4 cookies? And 4 different cookies? (Answer: 126, 15.)

8. In a school, there are the options of studying either Spanish and French. The 90% of the students are coursing Spanish while the rest course French. Among those that study Spanish 30% are boys while among those that study French, 40% are boys. Randomly choosing a student, compute the probability that is a girl. (Answer: 0.69.)

9. The 76% of students of Civil Engineering have failed Materials, and 45% have failed Statics. Moreover, 30% of them failed both courses. If a randomly selected student has failed Materials, what is the probability that he has as well failed Statics? (Answer: 0.39.)

10. The probability that a randomly selected person likes ice cream is 60%, while the probability a person likes pancakes is 36%. Moreover, the probability that a person likes pancakes given that the individual likes ice cream is 40%. Compute the probability that a person likes ice cream, given that the individual likes pancakes. (Answer: 0.667.)

3 Discrete Random Variables (Chapter 3)

Definition 3.1 *A random variable is a function whose domain is the sample space and whose range is the real numbers.*

Remark 3.2 *Roughly speaking, a random variable is any rule that associates a number with each possible outcome.*

Example 3.3 *Consider the sample space $\mathcal{S} = \{S, F\}$ be composed by S (success) and F (failure). We can define a random variable X by*

$$X(S) = 1\,, \qquad\qquad X(F) = 0\,.$$

Definition 3.4 *Any random variable whose only possible values are 0 and 1 is called a Bernoulli random variable.*

Definition 3.5 *A discrete random variable is a random variable whose possible values are a discrete set (i.e., a countably infinite set).*

Example 3.6 *The number of unbroken eggs in a randomly chosen standard egg carton, is a discrete random variable.*

3.1 Probability Distributions

Definition 3.7 *The probability distribution or probability mass function of a discrete random variable X is defined by*

$$p(x) = P(X = x)\,,$$

for every number x. In other words, $p(x)$ is the probability of all events $A \in \mathcal{S}$ such that $X(A) = x$.

Example 3.8 *Consider six boxes of components. In each box there may be a number of defective components, as follows: Box 1, 3, and 6 have no defective components; Box 4 has 1 defective component; and, Box 2 and 5 have 2 defective components. We define the discrete random variable X to be the number of defective components in the corresponding box, i.e.,*

$$\begin{aligned} X(\text{Box } 1) &= 0\,, \quad X(\text{Box } 2) = 2\,, \quad X(\text{Box } 3) = 0\,, \\ X(\text{Box } 4) &= 1\,, \quad X(\text{Box } 5) = 2\,, \quad X(\text{Box } 6) = 0\,. \end{aligned}$$

One of the boxes is randomly selected for shipment. Compute the probability of each event. (Answer: $p(0) = 0.5$, $p(1) = 1/6$ and $p(2) = 1/3$.)

Example 3.9 *At a certain electronic store they buy computers, either laptops or desktops. Define the discrete random variable X such that $X = 1$ if the costumer buys a desktop computer and $X = 0$ if the costumer buys a laptop. If 20% of all purchasers select a desktop, compute the probability mass function. (Answer: $p(0) = 0.8$ and $p(1) = 0.2$.)*

Definition 3.10 *The cumulative distribution function $F(x)$ of a discrete random variable X with probability mass function $p(x)$ is defined by*

$$F(x) = P(X \leq x) = \sum_{y \leq x} p(y)\,,$$

for every number x.

Remark 3.11 *For any value x, $F(x)$ is the probability that the observed value of X will be at most x.*

Example 3.12 *A store carries flash drives with either 1 GB, 2 GB, 4 GB, 8 GB, or 16 GB of memory. The probability mass function $p(x)$ is given by*

$$p(1) = 0.05\,, \quad p(2) = 0.1\,, \quad p(4) = 0.35\,, \quad p(8) = 0.4\,, \quad p(16) = 0.1\,.$$

Compute the cumulative distribution function.

Proposition 3.13 *For any two numbers a and b such that $a \leq b$, then*

$$P(a \leq X \leq b) = F(b) - F(a^*)\,,$$

where a^* represents the largest possible X value that is strictly less than a.

Example 3.14 *Consider the discrete random variable X to be the number of days of sick leave taken by a randomly selected employee during a particular year. If the maximum number of allowable sick days per year is 14, then the possible values of X are 0,...,14. Assume $F(0) = 0.58$, $F(1) = 0.72$, $F(2) = 0.76$, $F(3) = 0.81$, $F(4) = 0.88$ and $F(5) = 0.94$.*

1. *Compute $P(2 \leq X \leq 5)$. (Answer: 0.22.)*
2. *Compute $P(X = 3)$. (Answer: 0.05.)*

3.2 Expected Values

Definition 3.15 *Let X be a discrete random variable of possible values D and probability mass function $p(x)$. The expected value of X, denoted by $E(X)$ (also, by μ_X) is*

$$E(X) = \sum_{x \in D} x p(x)\,.$$

Example 3.16 *Let X be a Bernoulli random variable with probability mass function $p(1) = p$ and $p(0) = 1 - p$. Then,*

$$E(X) = 0p(0) + 1p(1) = p\,.$$

Proposition 3.17 *Let X be a discrete random variable. Then for any function h, $h(X)$, is also a discrete random variable and*

$$E(h(X)) = \sum_{x \in D} h(x)p(x)\,.$$

Example 3.18 *A computer store has purchased three computers of a certain type at \$500 apiece. It will sell them from \$1000 apiece. The manufacturer has agreed to repurchase any computers still unsold after a specified period at \$200 apiece. Let X denote the number of computers sold, and suppose $p(0) = 0.1$, $p(1) = 0.2$, $p(2) = 0.3$ and $p(3) = 0.4$. With $h(X)$ denoting the profit associated with selling X units, the given information implies that*

$$h(X) = 1000X + 200(3 - X) - 1500 = 800X - 900\,.$$

Then, the expected profit is then:

$$E(h(X)) = h(0)p(0) + h(1)p(1) + h(2)p(2) + h(3)p(3) = 700\,.$$

Proposition 3.19 *Let h be an affine function, $h(X) = aX + b$. Then,*

$$E(h(X)) = E(aX + b) = aE(X) + b\,.$$

Definition 3.20 *Let X be a discrete random variable with probability mass function $p(x)$ and expected value μ. Then the* <u>*variance*</u> *of X, denoted by $V(X)$ (also by σ_X^2) is*

$$V(X) = \sum_{x \in D} (x - E(X))^2 p(x) = E\left((X - E(X))^2\right)\,.$$

The <u>*standard deviation*</u> *of X is $\sqrt{V(X)}$, often denoted by σ_X.*

Example 3.21 *A library has an upper limit of 6 on the number of DVDs that can be checked out to an individual at one time. Consider only those who currently have DVDs checked out and let X denote the number of DVDs checked out to a randomly selected individual. Assume that the probability mass function of X is given by*

$$p(1) = 0.3\,, \quad p(2) = 0.25\,, \quad p(3) = 0.15\,, \quad p(4) = 0.05\,, \quad p(5) = 0.1\,, \quad p(6) = 0.15\,.$$

The expected value of X is $E(X) = 2.85$, while the variance is $V(X) = 3.2275$.

Proposition 3.22 *Let X be a discrete random variable. Then*

$$V(X) = E(X^2) - E(X)^2\,.$$

Proposition 3.23 *Let X be a discrete random variable. Then,*

$$V(aX + b) = a^2 V(X)\,,$$

for any real numbers a and b.

3.3 The Binomial Probability Distribution

Definition 3.24 *Assume that the following conditions are satisfied for an experiment:*

(i) The experiment consists of a sequence of n fixed smaller experiments, called trials.

(ii) Each trial can result in either success or failure. We say that these trials are dichotomous trials.

(iii) The trials are independent.

(iv) The probability of success is constant from trial to trial.

Then, we say that the experiment is a binomial experiment.

Example 3.25 *Toss a coin n times in a raw.*

Definition 3.26 *The binomial random variable $X \sim \mathrm{Bin}(n,p)$ associated with a binomial experiment consisting of n trials with probability of success p is defined as the number of successes among the n trials.*

Theorem 3.27 *The probability mass function of a binomial random variable $X \sim \mathrm{Bin}(n,p)$ is given by*

$$b(x;n,p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}\,, & x = 0, ..., n\,, \\ 0\,, & \textit{otherwise}\,. \end{cases}$$

Moreover, the cumulative distribution function is

$$B(x;n,p) = P(X \leq x) = \sum_{y=0}^{x} b(y;n,p)\,,$$

for $x = 0, ..., n$.

Example 3.28 *Each of six randomly selected soda drinkers is given a glass containing regular soda (S) and one containing a diet soda (F). Suppose there is actually no tendency among the drinkers to prefer one to the other. Then:*

1. *Compute the probability p of a selected individual preferring regular soda. (Answer: $p = 0.5$).*

2. *Let X be the number among the six who prefer regular soda, then $X \sim \mathrm{Bin}(6,p)$. Compute the probability that at least three prefer regular soda. (Answer: $P(X \geq 3) = \sum_{x=3}^{6} b(x;6,1/2) = 0.656$).*

3. *Compute the probability that at most one prefers regular soda. (Answer: $P(X \leq 1) = 0.109$).*

Proposition 3.29 *If $X \sim \mathrm{Bin}(n,p)$, then $E(X) = np$ and $V(X) = np(1-p)$.*

3.4 The Poisson Probability Distribution

Definition 3.30 *A discrete random variable X is said to have a* <u>*Poisson distribution*</u> *with parameter $\mu > 0$ if the probability mass function of X is*

$$p(x;\mu) = e^{-\mu}\frac{\mu^x}{x!},$$

for $x = 0, 1, 2, \ldots..$

Proposition 3.31 *Suppose that in the binomial probability mass function $b(x;n,p)$ we let the number of trials $n \to \infty$ and $p \to 0$ such that $np \to \mu > 0$. Then $b(x;n,p) \to p(x;\mu)$.*

Example 3.32 *Let X denote the number of defects in a particular transistor, and suppose it has a Poisson distribution with $\mu = 2$.*

1. *Compute the probability that there are exactly three defects. (Answer: $P(X = 3) = p(3,2) = e^{-2}\frac{2^3}{3!} = 0.18$).*
2. *Compute the probability that there are at most three defects. (Answer: $P(X \leq 3) = \sum_{x=0}^{3} e^{-2}\frac{2^x}{x!} = 0.857$.)*

Proposition 3.33 *If X has a Poisson distribution with parameter μ, then $E(X) = V(X) = \mu$.*

4 Continuous Random Variables (Chapter 4)

Definition 4.1 *A continuous random variable is a random variable whose possible values are an uncountable set and no possible value of the variable has positive probability.*

Definition 4.2 *Let X be a continuous random variable. Then a probability distribution or probability density function of X is a function $f(x)$ such that for any two numbers $a \leq b$,*

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx\,.$$

Remark 4.3 *The probability that X takes on a value in the interval $[a, b]$ is the area below the density function.*

Definition 4.4 *The cumulative distribution function $F(x)$ for a continuous random variable X is the function*

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\, dy\,,$$

for every number $x \in \mathbb{R}$.

Remark 4.5 *For every x, $F(x)$ represents the area below the density function to the left of x. $F(x)$ increases smoothly as x increases.*

Proposition 4.6 *Let X be a continuous random variable with probability density function $f(x)$ and cumulative distribution function $F(x)$. Then, for any number $a \in \mathbb{R}$,*

$$P(X > a) = 1 - F(a)\,,$$

and for any two numbers $a < b$,

$$P(a \leq X \leq b) = F(b) - F(a)\,.$$

Definition 4.7 *The expected or mean value of a continuous random variable X with probability density function $f(x)$ is*

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx\,.$$

Proposition 4.8 *If X is a continuous random variable with probability density function $f(x)$ and $h(X)$ is any function of X, then*

$$E[h(X)] = \mu_{h(X)} = \int_{-\infty}^{\infty} h(x) f(x)\, dx\,.$$

Definition 4.9 *The* *variance* *of a continuous random variable* X *with probability density function* $f(x)$ *and mean value* μ_X *is*

$$\sigma_X^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x)\, dx = E\left((X - \mu_X)^2\right) .$$

The *standard deviation* *of* X *is* $\sigma_X = \sqrt{V(X)}$.

Proposition 4.10 *Let* X *be a continuous random variable. Then,*

$$V(X) = E(X^2) - [E(X)]^2 .$$

4.1 The Uniform Distribution

Definition 4.11 *A continuous random variable* X *is said to have a* *uniform distribution* *on the interval* $[a, b]$ *if the probability density function of* X *is*

$$f(x; a, b) = \begin{cases} \frac{1}{b-a} , & a \leq x \leq b , \\ 0 , & \textit{otherwise} . \end{cases}$$

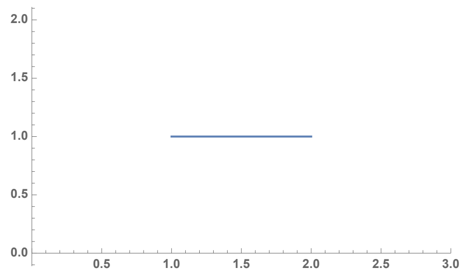


Figure 1: A uniform distribution on the interval $[1, 2]$.

Example 4.12 *Consider the reference line connecting the valve stem on a tire to the center point, and let* X *be the angle measured clockwise to the location of an imperfection. One possible probability density function for* X *is*

$$f(x) = \begin{cases} 1/360 , & 0 \leq x < 360 , \\ 0 , & \textit{otherwise} . \end{cases}$$

Compute the probability that the angle of occurrence is within 90 *of the reference line.*

4.2 The Normal Distribution

Definition 4.13 *A continuous random variable X is said to have a* normal distribution *with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ if the probability density function of X is*

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\ \sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

for $x \in \mathbb{R}$.

Remark 4.14 *The density curve associated to a normal distribution is a Gaussian bell. The parameter μ is the* location parameter, *and it gives the value where the curve is centered. The parameter σ is the* scale parameter *and changing its value stretches or compresses the curve.*

Definition 4.15 *The normal distribution with parameter value $\mu = 0$ and $\sigma = 1$ is called the* standard normal distribution. *A continuous random variable having a standard normal distribution is called a* standard normal random variable *and will be denoted by Z.*

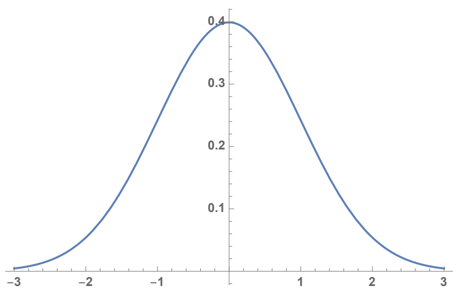


Figure 2: The standard normal distribution.

Example 4.16 *Compute the following probabilities:*

1. $P(Z \leq 1.25)$. *(Answer:* 0.8944*).*
2. $P(Z > 1.25)$. *(Answer:* 0.1056*).*
3. $P(Z \leq -1.25)$. *(Answer:* 0.1056*).*
4. $P(-0.38 \leq Z \leq 1.25)$. *(Answer:* $P(Z \leq 1.25) - P(Z \leq -0.38) = 0.8944 - 0.3520 = 0.5424$*).*

Remark 4.17 *A very useful critical value is $z = 1.645$, for which $P(Z \leq 1.645) = 0.95$, that is a 95%.*

Remark 4.18 *Although the standard normal distribution almost never serves as a model for a naturally arising population, it is really useful to obtain information about the more general normal distributions. Indeed, there exist tables with the corresponding probabilities tabulated (see Appendix A).*

Theorem 4.19 *If X has a normal distribution with mean μ and standard deviation σ, then $Z = (X - \mu)/\sigma$ has a standard normal distribution. Therefore,*

$$P(a \leq X \leq b) = P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right).$$

Example 4.20 *Assume that reaction time for an in-traffic response to a brake signal from standard break lights can be modeled with a normal distribution having mean value 1.25 seconds and standard deviation of 0.46 seconds. What is the probability that reaction time is between 1.00 seconds and 1.75 seconds? (Answer: $P(1.00 \leq X \leq 1.75) = P(-0.54 \leq Z \leq 1.09) = 0.8621 - 0.2946 = 0.5675$).*

Example 4.21 *The breakdown voltage of a randomly chosen diode of a particular type is known to be normally distributed. What is the probability that a diode's breakdown voltage is within 1 standard deviation of its mean value? (Answer: $P(\mu - \sigma \leq X \leq \mu + \sigma) = P(-1 \leq Z \leq 1) = 0.6826$).*

Remark 4.22 *If the population distribution of a variable is normal, then about 68% of the values are within 1 standard deviation of the mean, and about 95% within 2.*

4.3 The Exponential Distribution

Definition 4.23 *A continuous random variable is said to have an exponential distribution with (scale) parameter $\lambda > 0$ if the probability density function of X is*

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 4.24 *Let X be a continuous random variable with exponential distribution of parameter $\lambda > 0$. Then the mean value and the variance are, respectively,*

$$\mu = \frac{1}{\lambda} \qquad \sigma^2 = \frac{1}{\lambda^2}.$$

Proposition 4.25 *Let X be a continuous random variable with exponential distribution. Then, the cumulative distribution function is*

$$F(x;\lambda) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$

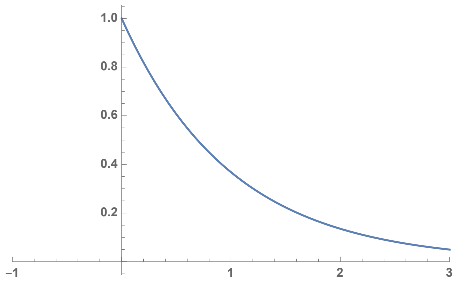


Figure 3: The exponential distribution with scale parameter $\lambda = 1$.

Example 4.26 *Assume that the distribution of stress range in certain bridge connections follows the exponential distribution with mean value* 6*. Compute:*

1. *The probability that stress range is at most* 10*. (Answer:* $P(X \leq 10) = F(10; 0.1667) = 0.811$*.)*
2. *The probability that stress range is between* 5 *and* 10*. (Answer:* $P(5 \leq X \leq 10) = 0.246$*.)*

Remark 4.27 *Suppose that the number of events occurring in any time interval of length t has a Poisson distribution with parameter αt (where α, the rate of the event process, is the expected number of events occurring in* 1 *unit of time) and that numbers of occurrences in non-overlapping intervals are independent of one another. Then the distribution of elapsed time between the occurrence of two successive events is exponential with parameter $\lambda = \alpha$.*

Example 4.28 *Suppose that calls to a certain center occur according to a Poisson process with rate $\alpha = 0.5$ call per day. (Then the number of days X between successive calls has an exponential distribution with parameter* 0.5*.) Compute the probability that more than* 2 *days elapse between calls. (Answer:* $P(X > 2) = 1 - F(2; 0.5) = 0.368$*.)*

Remark 4.29 *The exponential distribution is usually used to model the distribution of component lifetime. A partial reason is that the exponential distribution is said to have "memoryless" property. Suppose component lifetime is exponentially distributed with parameter λ. After putting the component into service, we leave for a period t_o and then return to find the component still working. What is now the probability that it lasts at least and additional t period?*

$$P(X \geq t \,|\, X \geq t_o) = \frac{P\left((X \geq t + t_o) \cap (X \geq t_o)\right)}{P(X \geq t_o)}\,.$$

The event $X \geq t_o$ is redundant. Therefore,

$$P(X \geq t \,|\, X \geq t_o) = \frac{P(X \geq t + t_o)}{P(X \geq t_o)} = e^{-\lambda t}\,.$$

Hence, this conditional probability is identical to the original probability $P(X \geq t)$. Roughly speaking, the distribution of additional lifetime is exactly the same as the original distribution of lifetime.

4.4 The Gamma Distribution

Definition 4.30 *For every $\alpha > 0$, the gamma function $\Gamma(\alpha)$ is defined by*

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx\,.$$

Proposition 4.31 *The following properties hold for the gamma function $\Gamma(\alpha)$:*

(i) *For any $\alpha > 1$, $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$ (which follows integrating by parts).*

(ii) *For any positive integer n, $\Gamma(n) = (n-1)!$.*

(iii) $\Gamma(1/2) = \sqrt{\pi}$.

Definition 4.32 *A continuous random variable X is said to have a gamma distribution if the probability density function of X is*

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\,, & x \geq 0\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

where the parameters α and β satisfy $\alpha > 0$ and $\beta > 0$. The standard gamma distribution has $\beta = 1$.

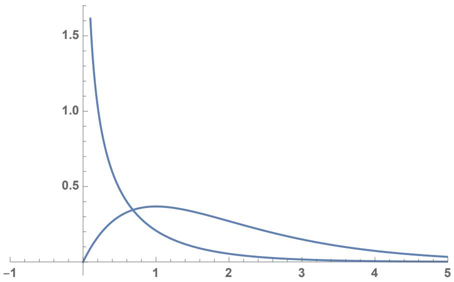


Figure 4: Two gamma distributions for the parameters $\alpha = 1/2$ and $\alpha = 2$, respectively, and $\beta = 1$.

Proposition 4.33 *The mean and variance of a continuous random variable X having the gamma distribution are, respectively,*

$$\mu = \alpha\beta\,, \qquad \sigma^2 = \alpha\beta^2\,.$$

Example 4.34 *The gamma distribution is widely used to model the extent of degradation such as corrosion, creep, or wear. Let X represent the amount of degradation of a certain type, and suppose that it has a standard gamma distribution with $\alpha = 2$. Compute:*

1. *The probability that the amount of degradation is between 3 and 5. (Answer: $P(3 \leq X \leq 5) = F(5;2) - F(3;2) = 0.960 - 0.801 = 0.159$.)*

2. *The probability that the amount of degradation exceeds 4. (Answer: $P(X > 4) = 1 - P(X \leq 4) = 1 - F(4;2) = 1 - 0.908 = 0.092$.*

4.5 The χ^2 Distribution

Definition 4.35 *Let ν be a positive integer. Then a continuous random variable X is said to have a χ^2 distribution with parameter ν if the probability density function of X is the gamma density with $\alpha = \nu/2$ and $\beta = 2$. The parameter ν is called the number of degrees of freedom.*

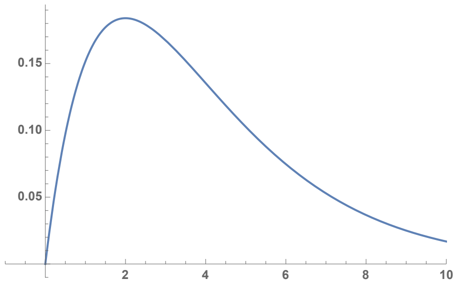


Figure 5: The χ^2 distribution with 2 degrees of freedom.

4.6 The Weibull Distribution

Definition 4.36 *A continuous random variable X is said to have a Weibull distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ if the probability density function of X is*

$$f(x;\alpha,\beta) = \begin{cases} \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-(x/\beta)^\alpha}\,, & x \geq 0\,, \\ 0\,, & x < 0\,. \end{cases}$$

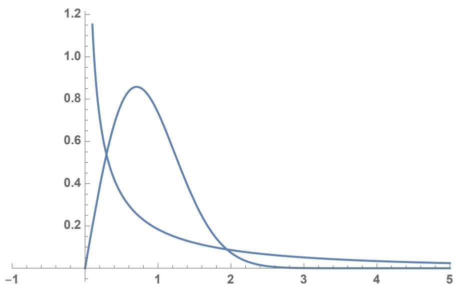


Figure 6: Two Weibull distributions for shape parameters $\alpha = 1/2$ and $\alpha = 2$, respectively, and scale parameters $\beta = 1$.

Proposition 4.37 *Assume X is a continuous random variable which has a Weibull distribution with shape parameter α and scale parameter β. Then, the mean and variance of X are, respectively,*

$$\mu = \beta\,\Gamma\left(1 + 1/\alpha\right)\,, \qquad \sigma^2 = \beta^2\left(\Gamma(1+2/\alpha) - [\Gamma(1+1/\alpha)]^2\right).$$

Moreover, the cumulative distribution function is

$$F(x;\alpha,\beta) = \begin{cases} 0\,, & x < 0\,, \\ 1 - e^{-(x/\beta)^{\alpha}}\,, & x \geq 0\,. \end{cases}$$

Example 4.38 *In recent years the Weibull distribution has been used to model engine emissions of various pollutants. Let X denote the amount of NO_x emission (g/gal) from a randomly selected four-stroke engine of a certain type, and suppose that X has a Weibull distribution with $\alpha = 2$ and $\beta = 10$. Compute:*

1. $P(X \leq 10)$. *(Answer: $P(X \leq 10) = F(10; 2, 10) = 1 - e^{-(10/10)^2} = 0.632$.)*
2. $P(X \leq 25)$. *(Answer: 0.998.)*
3. *The value c which separates the 5% of all engines having the largest amounts of NO_x emissions from the remaining 95%. (Answer: $0.95 = 1 - e^{-(c/10)^2}$, so $c = 17.3$.)*

Remark 4.39 *A Weibull model may also allow shifts in the smallest value (translating the curve). In this case, the cumulative distribution function is obtained by replacing x by $x - \gamma$.*

Example 4.40 *An understanding of the volumetric properties of asphalt is important in designing mixtures which will result in high-durability pavement. For a particular mixture, let X be the air void volume (%) and assume is modeled with a three-parameter Weibull distribution. Suppose the values of the parameters are $\gamma = 4$, $\alpha = 1.3$, and $\beta = 0.8$. Compute the probability that air void volume of a specimen is between 5% and 6%. (Answer: $P(5 \leq X \leq 6) = F(6; 1.3, 0.8, 4) - F(5; 1.3, 0.8, 4) = 1 - e^{-[(6-4)/0.8]^{1.3}} - (1 - e^{-[(5-4)/0.8]^{1.3}}) = 0.263 - 0.037 = 0.226$.)*

4.7 The Lognormal Distribution

Definition 4.41 *A nonnegative continuous random variable X is said to have a lognormal distribution if the random variable $Y = \log(X)$ has a normal distribution. The resulting probability density function of a lognormal random variable when $\log(X)$ is normally distributed with parameters μ and σ is*

$$f(x;\mu,\sigma) = \begin{cases} \frac{1}{\sqrt{2\pi}\,\sigma x} e^{-[\log(x)-\mu]^2/(2\sigma^2)}\,, & x \geq 0\,, \\ 0\,, & x < 0\,. \end{cases}$$

Proposition 4.42 *Let X be a continuous random variable X lognormally distributed. The mean and variance of X are, respectively,*

$$E(X) = e^{\mu + \sigma^2/2}\,, \qquad V(X) = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}\,.$$

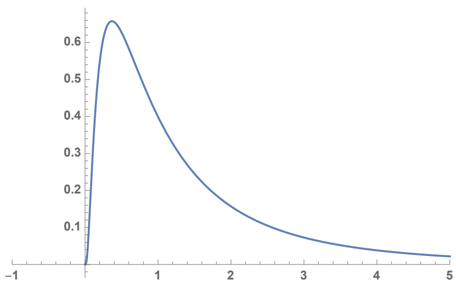


Figure 7: The lognormal distribution for $\mu = 0$ and $\sigma = 1$.

Proposition 4.43 *The cumulative distribution function of a nonnegative continuous random variable X with lognormal distribution is given by*

$$F(x; \mu, \sigma) = P(X \leq x) = P(\log(X) \leq \log(x)) = P\left(Z \leq \frac{\log(x) - \mu}{\sigma}\right).$$

Example 4.44 *The lognormal distribution seems to be the best option for describing the distribution of maximum pit depth data from cast iron pipes in soil. Assume that a lognormal distribution with $\mu = 0.353$ and $\sigma = 0.754$ is appropriate for maximum pit depth (mm) of buried pipelines. Compute:*

1. *The mean and variance of the pit depth. (Answer: $E(X) = 1.891$ and $V(X) = 2.7387$.)*
2. *The probability that maximum pit depth is between 1 and 2 mm. (Answer: $P(1 \leq X \leq 2) = 0.354$.)*
3. *What value c is such that only 1% have a maximum pit depth exceeding c? (Answer: $0.99 = P(X \leq c)$ and so $c = 8.247$.)*

4.8 The Beta Distribution

Definition 4.45 *A continuous random variable X is said to have a beta distribution with parameters $\alpha > 0$, $\beta > 0$, A and B if the probability density function of X is*

$$f(x; \alpha, \beta, A, B) = \begin{cases} \frac{1}{B-A} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{x-A}{B-A}\right)^{\alpha-1} \left(\frac{B-x}{B-A}\right)^{\beta-1}, & A \leq x \leq B, \\ 0, & \textit{otherwise}. \end{cases}$$

Proposition 4.46 *Let X be a continuous random variable X having a beta distribution. The mean and variance of X are, respectively,*

$$\mu = A + (B - A)\frac{\alpha}{\alpha + \beta}, \qquad \sigma^2 = \frac{(B - A)^2 \alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

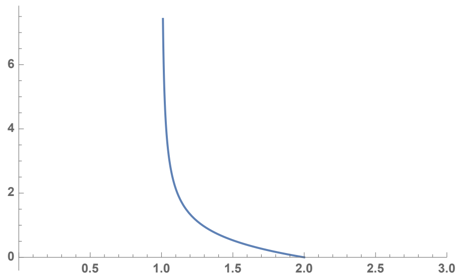


Figure 8: The beta distribution with parameters $\alpha = 1/2$, $\beta = 2$, $A = 1$ and $B = 2$.

Example 4.47 *Project managers often use a method labeled PERT (for Program Evaluation and Review Technique) to coordinate the various activities making up a large project. One successful application was in the construction of the Apollo spacecraft. A standard assumption in PERT analysis is that the time necessary to complete any particular activity once it has been started has a beta distribution with A being the optimistic time and B the pessimistic one. Suppose that in constructing a single-family house, the time X (in days) necessary for lying the foundation has a beta distribution with $A = 2$, $B = 5$, $\alpha = 2$ and $\beta = 3$. Compute:*

1. *The mean of X. (Answer: 3.2.)*
2. *The probability that it takes at most 3 days to lay the foundation. (Answer: $P(X \leq 3) = 0.407$.)*

4.9 The Student's t Distribution

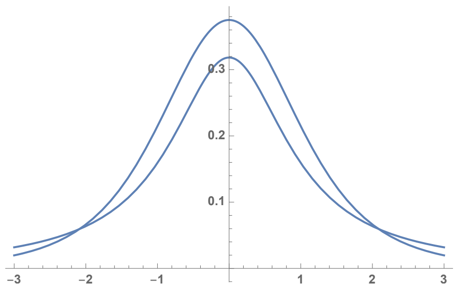


Figure 9: Two t distributions with parameters $\nu = 1$ and $\nu = 4$, respectively.

Definition 4.48 *A continuous random variable X is said to have a* <u>*(Student's) t distribution*</u> *with parameter $\nu > 0$, if the probability density function of X is*

$$f(x;\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\,\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}},$$

for every $x \in \mathbb{R}$. The parameter ν is called the <u>*degrees of freedom*</u>*. (See Appendix B.)*

Review Problems

1. A factory uses three production lines to manufacture cans of a certain type. In the following lines we give the percentages of nonconforming cans, categorized by type of nonconformance, for each of the three lines during a particular time period:

   Blemish: Line 1, 15; Line 2, 12; and, Line 3, 20.

   Crack: Line 1, 50; Line 2, 44; and, Line 3, 40.

   Pull-Tab Problem: Line 1, 21; Line 2, 28; and, Line 3, 24.

   Surface Defect: Line 1, 10; Line 2, 8; and, Line 3, 15.

   Other: Line 1, 4; Line 2, 8; and, Line 3, 2.

   During this period, line 1 produced 500 nonconforming cans, line 2 produced 400 such cans, and line 3 was responsible for 600 nonconforming cans. Suppose that one of these 1500 cans is randomly selected.

   (a) What is the probability that the can was produced by line 1? (Answer: 1/3.)

   (b) If the selected can came from line 1, what is the probability that it had a blemish? (Answer: 0.150.)

   (c) Given that the selected can had a surface defect, what is the probability that it came from line 1? (Answer: 0.291.)

2. Individual A has a circle of five close friends (B, C, D, E and F). A has heard a certain rumor from outside the circle and has invited the five friends to a party to circulate the rumor. To begin, A selects one of the five at random and tells the rumor to the chosen individual. That individual then selects at random one of the four remaining individuals and repeats the rumor. Continuing, a new individual is selected from those not already having heard the rumor by the individual who has just heard it, until everyone has been told.

   (a) What is the probability that the rumor is repeated in the order B, C, D, E and F? (Answer: 0.083.)

   (b) What is the probability that F is the third person at the party to be told the rumor? (Answer: 0.2.)

   (c) What is the probability that F is the last person to hear the rumor? (Answer: 0.2.)

   (d) If at each state the person who currently "has" the rumor does not know who has already heard it and selects the next recipient at random from all five possible individuals, what is the probability that F has still not heard the rumor after it has been told ten times at the party? (Answer: 0.1074.)

3. Of all customers purchasing automatic garage-door openers, 75% purchase a chain-driven model. Let X be the number among the next 15 purchasers who select the chain-driven model.

   (a) What is the probability mass function of X? (Answer: $b(x; 15, 0.75)$.)

   (b) Compute $P(X \geq 10)$. (Answer: 0.686.)

   (c) Compute $P(6 \leq X \leq 10)$. (Answer: 0.313.)

   (d) Compute μ and σ^2. (Answer: 11.25, 2.81.)

   (e) If the store currently has in stock 10 chain-driven models and 8 shaft-driven models, what is the probability that the requests of these 15 customers can all be met from existing stock? (Answer: 0.310.)

4. Suppose that the number of drivers who travel between a particular origin and destination during a designated time period has a Poisson distribution with parameter $\mu = 20$. What is the probability that the number of drivers will:

   (a) Be at most 10? (Answer: 0.011.)

   (b) Exceed 20? (Answer: 0.441.)

   (c) Be between 10 and 20, inclusive? Be strictly between 10 and 20? (Answer: 0.554, 0.459.)

   (d) Be within 2 standard deviations of the mean value? (Answer: 0.945.)

5. The Weibull distribution is proposed as a model for time (in hours) to failure of solid insulating specimens subjected to AC voltage. The values of the parameters depend on the voltage and temperature; suppose $\alpha = 2.5$ and $\beta = 200$.

   (a) What is the probability that a specimen's lifetime is at most 250? Less than 250? More than 300? (Answer: 0.826, 0.826, 0.0636.)

   (b) What is the probability that a specimen's lifetime is between 100 and 250? (Answer: 0.664.)

   (c) What value is such that exactly 50% of all specimens have lifetime exceeding that value? (Answer: 172.727.)

6. Let X be the nonpoint source load of total dissolved solids, which can be modeled with a lognormal distribution having mean value 10281 kg/km/day and coefficient of variation (σ_X/μ_x) of 0.4.

   (a) What are the mean value and standard deviation of $\log(X)$? (Answer: 9.164, 0.385.)

   (b) What is the probability that X is at most 15000 kg/day/km? (Answer: 0.879.)

(c) What is the probability that X exceeds its mean value, and why is this probability not 0.5? (Answer: 0.4247.)

(d) Is 17000 the 95th percentile of distribution? (Answer: No, $P(Z \leq 1.498) = 0.9332$.)

7. The defect length of a corrosion defect in a pressurized steel pipe is normally distributed with mean value 30mm and standard deviation 7.8mm.

   (a) What is the probability that defect length is at most 20mm? Less than 20mm? (Answer: 0.1003.)

   (b) What is the 75th percentile of defect length distribution? (Answer: 35.226.)

   (c) What is the 15th percentile of the defect length distribution? (Answer: 21.888.)

   (d) What values separate the middle 80% of the defect length distribution from the smallest 10% and largest 10%? (Answer: 20.016 and 39.984.)

5 Point Estimation (Chapter 6)

Definition 5.1 *A point estimate of a parameter θ is a single number that can be regarded as a sensible value for θ. It is obtained by selecting a suitable statistic and computing its value from the given sample data. The selected statistic is called the point estimator of θ.*

Example 5.2 *A random sample of $n = 3$ batteries might yield observed lifetimes (hours) $x_1 = 5$, $x_2 = 6.4$ and $x_3 = 5.9$. The computed value of the sample mean lifetime is $\bar{x} = 5.77$, and it is reasonable to regard 5.77 as a plausible value of μ, the population mean.*

Definition 5.3 *A point estimator $\widehat{\theta}$ is said to be an unbiased estimator of θ if $E(\widehat{\theta}) = \theta$ for every possible value of θ. If $\widehat{\theta}$ is not unbiased, the difference $E(\widehat{\theta}) - \theta$ is called the bias of $\widehat{\theta}$.*

Remark 5.4 *It may seem necessary to know the value of θ to see whether $\widehat{\theta}$ is unbiased or not. This is not usually the case though since unbiasedness is a general property of the estimator's sampling distribution (where it is centered) which typically does not depend on any particular parameter value.*

Proposition 5.5 *When X is a binomial random variable with parameters n and p, the sample proportion $\widehat{p} = X/n$ is an unbiased estimator of p.*

Example 5.6 *Suppose that X, the reaction time to a certain stimulus, has a uniform distribution on the interval from 0 to an unknown upper limit θ. It is desired to estimate θ on the basis of a random sample $X_1, ..., X_n$ of reaction times. Since θ is the largest possible time in the entire population of reaction times, consider as a first estimator the largest sample reaction time: $\widehat{\theta}_1 = \max\{X_1, ..., X_n\}$. For instance, if $n = 5$ and $x_1 = 4.2$, $x_2 = 1.7$, $x_3 = 2.4$, $x_4 = 3.9$ and $x_5 = 1.3$, the point estimate of θ is $\widehat{\theta}_1 = 4.2$. This will always be a biased estimate because*

$$E(\widehat{\theta}_1) = \frac{n}{n+1}\theta < \theta\,.$$

However, we can modify $\widehat{\theta}_1$ to get an unbiased estimator, namely,

$$\widehat{\theta}_2 = \frac{n+1}{n}\max\{X_1, ..., X_n\}\,.$$

Theorem 5.7 (Principle of Unbiased Estimation) *When choosing among several different estimators of θ, select one that is unbiased.*

Proposition 5.8 *If $X_1, ..., X_n$ is a random sample from a distribution with mean μ, then $\bar{X}$ is an unbiased estimator of μ. If in addition the distribution is continuous and symmetric, then $\widetilde{X}$ and any trimmed mean are also unbiased estimators of μ.*

Proposition 5.9 *Let $X_1, ..., X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then the estimator*

$$\widehat{\sigma}^2 = S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2 ,$$

is unbiased for estimating σ^2.

Remark 5.10 *There are examples in which there exist several unbiased estimators for a particular parameter.*

Theorem 5.11 (Principle of Minimum Variance Unbiased Estimation) *Among all estimators of θ that are unbiased, choose the one that has a minimum variance. The resulting $\widehat{\theta}$ is called the minimum variance unbiased estimator of θ.*

Example 5.12 *The estimator $\widehat{\theta}_2$ of previous example was an unbiased estimator, but it is not the only one. For instance, since the expected value of a uniformly distributed random variable is just the midpoint of the interval, then $E(X_i) = \theta/2$ (this follows from the definition) and, hence, $E(2\bar{X}) = \theta$. That is, the estimator $\widehat{\theta}_3 = 2\bar{X}$ is also unbiased for θ. Nevertheless, the estimator $\widehat{\theta}_2$ has smaller variance. We use:*

$$V(X_i) = \sigma^2 = E(X_i^2) - E(X_i)^2 = \int_0^\theta \frac{x^2}{\theta}\, dx - \frac{\theta^2}{4} = \frac{\theta^2}{12} ,$$

and so $V(\bar{X}) = \theta^2/(12n)$, i.e., $V(2\bar{X}) = 4V(\bar{X}) = \theta^2/(3n)$. On the other hand, $V(\widehat{\theta}_2) = (n+1)^2\theta^2/(12n^2)$.

Theorem 5.13 *Let $X_1, ..., X_n$ be a random sample from a normal distribution with parameters μ and σ. Then the estimator $\widehat{\mu} = \bar{X}$ is the minimum value unbiased estimator for μ.*

Definition 5.14 *The standard error of an estimator $\widehat{\theta}$ is its standard deviation $\sigma_{\widehat{\theta}} = \sqrt{V(\widehat{\theta})}$.*

Remark 5.15 *If the standard error itself involves unknown parameters whose values can be estimated, substitution of these estimates into $\sigma_{\widehat{\theta}}$ yields the estimated standard error.*

6 Confidence Intervals (Chapter 7)

Remark 6.1 *Assume that the population distribution is normal. More precisely, the actual sample observations* $x_1, ..., x_n$ *are assumed to be the result of a random sample* $X_1, ..., X_n$ *from a normal distribution with mean value* μ *and standard deviation* σ*. Then, the sample mean* $\bar{X}$ *is normally distributed with expected value* μ *and standard deviation* $\sigma/\sqrt{n}$*. Then, we have the standard normal variable*

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\,.$$

Definition 6.2 *Let* $x_1, ..., x_n$ *be the observations from a normal distribution with mean value* μ *and standard deviation* σ*. The interval*

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right),$$

is called a <u>95% *confidence interval*</u> *for* μ*.*

Example 6.3 *Industrial engineers who specialize in ergonomics are concerned with designing workspace and worker-operated devices so as to achieve high productivity and comfort. In order to determine the preferred height for an experimental keyboard with large forearm-wrist support, a sample of* $n = 31$ *trained typists was selected, and the preferred keyboard height was determined for each typist. The resulting sample average preferred height was* $\bar{x} = 80$ *(cm). Assuming that the preferred height is normally distributed with* $\sigma = 2$ *(cm), compute the* 95% *confidence interval for* μ*, the true average preferred height for the population of all experienced typists. (Answer:* $(79.3, 80.7)$*.)*

Remark 6.4 *A confidence interval does not mean that the probability of* μ *lying in that interval is* 0.95*. The correct interpretation is that if we keep picking samples over and over again the* 95% *of times the average will lie in that interval.*

Remark 6.5 *Observe that the value* 1.96 *in above definition comes from the critical value* $c = 1.96$ *such that*

$$P\left(-c \leq Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq c\right) = 0.95\,,$$

which means that the area below the standard normal curve between -1.96 *and* 1.96 *is* 0.95*. However, we may as well pick other critical values to get other confidence intervals.*

Definition 6.6 *A* <u>$100(1-\alpha)\%$ *confidence interval*</u> *for the mean* μ *of a normal population when the value of* σ *is known is given by*

$$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right),$$

where $z_{\alpha/2}$ *is the critical value such that*

$$P\left(-z_{\alpha/2} \leq Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right) = 1 - \alpha\,.$$

Proposition 6.7 *The sample size for the confidence interval to have a width w is*

$$n = 4z_{\alpha/2}^2 \frac{\sigma^2}{w^2}\,.$$

Example 6.8 *Extensive monitoring of a computer time-sharing system has suggested that response time to a particular editing command is normally distributed with standard deviation 25 milisec. A new operating system has been installed, and we wish to estimate the true average response time μ for the new environment. Assuming that response times are still normally distributed with $\sigma = 25$, what sample size is necessary to ensure that the resulting 95% confidence interval has a width of (at most) 10? (Answer: $n = 96.04$ and so, since it must be a natural number, $n = 97$ is required.)*

Remark 6.9 *The confidence intervals described above rely heavily in the fact that the standard deviation σ is know, which is rarely the case.*

Theorem 6.10 *Let $\bar{X}$ be the mean of a random sample of size n from a normal distribution with mean μ. Then, the random variable*

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}\,,$$

has a t distribution with $n-1$ degrees of freedom.

Remark 6.11 *Recall that the probability density function of a t distribution with $n-1$ degrees of freedom is*

$$f(x; n-1) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{(n-1)\,\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \left(1 + \frac{x^2}{n-1}\right)^{-\frac{n}{2}}\,,$$

for every $x \in \mathbb{R}$.

Proposition 6.12 *Let $\bar{x}$ and s be the sample mean and sample standard deviation computed from the results of a random sample from a normal distributed population with mean μ. Then a $100(1-\alpha)\%$ confidence interval for μ is*

$$\left(\bar{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}\right),$$

where $t_{\alpha/2,n-1}$ is the t critical value, i.e., the value such that

$$P\left(-t_{\alpha/2,n-1} < T < t_{\alpha/2,n-1}\right) = 1 - \alpha\,.$$

Example 6.13 *The following is a data on the modulus of rupture of composite beams designed to add value to low-grade sweetgum lumbers:*

$$\begin{array}{llllll}
6807.99\,, & 7637.06\,, & 6663.28\,, & 6165.03\,, & 6991.41\,, & 6992.23\,, \\
6981.46\,, & 7569.75\,, & 7437.88\,, & 6872.39\,, & 7663.18\,, & 6032.28\,, \\
6906.04\,, & 6617.17\,, & 6984.12\,, & 7093.71\,, & 7659.50\,, & 7378.61\,, \\
7295.54\,, & 6702.76\,, & 7440.17\,, & 8053.26\,, & 8284.75\,, & 7347.95\,, \\
7422.69\,, & 7886.87\,, & 6316.67\,, & 7713.65\,, & 7503.33\,, & 7674.99\,.
\end{array}$$

The sample mean and sample standard deviation are 7203.191 *and* 543.54*, respectively. Compute a confidence interval for true average of the modulus of rupture using a confidence level of* 95%*. (Answer: The necessary* t *critical value is* $t_{0.025,29} = 2.045$ *and so* $(7000.253, 7406.129)$*.)*

Example 6.14 *Consider the following sample of fat content (in percentage) of* $n = 10$ *randomly selected hot dogs:*

$$25.2\,,\quad 21.3\,,\quad 22.8\,,\quad 17.0\,,\quad 29.8\,,\quad 21.0\,,\quad 25.5\,,\quad 16.0\,,\quad 20.9\,,\quad 19.5\,.$$

Assuming that these were selected from a normal population distribution, compute a 95% *confidence interval for the population mean fact content. (Answer: The sample mean is* $\bar{x} = 21.9$ *and the critical* t *value is* $t_{0.025,9} = 2.262$*, so* $(18.94, 24.86)$*.)*

Remark 6.15 *Suppose you are going to eat a single hot dog of this type and want a prediction for the resulting fat content. A point prediction, analogous to a point estimator is just* $\bar{x} = 21.9$*, but this does not give any information about the reliability or precision.*

Proposition 6.16 *A* *prediction interval* *for a single observation to be selected from a normal population distribution is*

$$\left(\bar{x} - t_{\alpha/2,n-1}s\sqrt{1+\frac{1}{n}}, \bar{x} + t_{\alpha/2,n-1}s\sqrt{1+\frac{1}{n}}\right).$$

The prediction level is $100(1-\alpha)\%$*.*

Example 6.17 *Compute the prediction interval for previous Exercise. (Answer:* $s = 4.134$ *and so* $(12.09, 31.71)$*.).*

Theorem 6.18 *Let* $X_1,\ldots,X_n$ *be a random sample from a normal distribution with parameters* μ *and* σ^2*. Then the random variable*

$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2,$$

has a χ^2 *probability distribution with* $n-1$ *degrees of freedom.*

Definition 6.19 *A* $100(1-\alpha)\%$ *confidence interval for the variance* σ^2 *of a normal population is*

$$\left((n-1)\frac{s^2}{\chi^2_{\alpha/2,n-1}}, (n-1)\frac{s^2}{\chi^2_{1-\alpha/2,n-1}}\right),$$

where

$$P\left(\chi^2_{1-\alpha/2,n-1} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{\alpha/2,n-1}\right) = 1-\alpha\,.$$

Example 6.20 *The accompanying data on breakdown voltage of electrically stressed circuits was obtained from a normal probability distribution:*

$$1470\,,\quad 1510\,,\quad 1690\,,\quad 1740\,,\quad 1900\,,\quad 2000\,,\quad 2030\,,\quad 2100\,,\quad 2190\,,$$
$$2200\,,\quad 2290\,,\quad 2380\,,\quad 2390\,,\quad 2480\,,\quad 2500\,,\quad 2580\,,\quad 2700\,.$$

Compute a 95% confidence interval for the variance of the breakdown voltage distribution. (Answer: The sample variance is $s^2 = 137324.3$, and the critical χ^2 values needed are $\chi^2_{0.975,16} = 6.908$ and $\chi^2_{0.025,16} = 28.845$. Then the interval is $(76172.3, 318064.4)$.)

Remark 6.21 *The confidence intervals described above rely heavily in the fact that the population is normally distributed. Without knowing this, but having sufficiently large sample, we can still find confidence intervals.*

Theorem 6.22 (Central Limit Theorem) *Let $X_1, ..., X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then, if n is sufficiently large, $\bar{X}$ has approximately a normal distribution with $\mu_{\bar{X}} = \mu$ and $\sigma^2_{\bar{X}} = \sigma^2/n$.*

Proposition 6.23 *If n is sufficiently large, the standardized variable*

$$Z = \frac{\bar{X} - \mu}{S/\sqrt{n}}\,,$$

where S is the sample standard deviation, has approximately a standard normal distribution. This implies that the interval

$$\left(\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}\right),$$

is a large-sample confidence interval for μ with confidence level approximately $100(1-\alpha)\%$.

Remark 6.24 *Above formula is valid regardless of the shape of the population distribution.*

Example 6.25 *Here are reported prices for a sample of 50 Boxsters, the cheapest model of Porsche:*

$$2948, 2996, 7197, 8338, 8500, 8759, 12710, 12925, 15767, 20000, 23247, 24863, 26000, 26210,$$
$$30552, 30600, 35700, 36466, 40316, 40596, 41021, 41234, 43000, 44607, 45000, 45027, 45442,$$
$$46963, 47978, 49518, 52000, 53334, 54208, 56062, 57000, 57365, 60020, 60265, 60803, 62851,$$
$$64404, 72140, 74594, 79308, 79500, 80000, 80000, 84000, 113000, 118634.$$

Compute the large-sample confidence interval for μ of about 95% confidence level. (Answer: $n = 50$, $\bar{x} = 45679.4$, $s = 26641.675$ and so $(38294.7, 53064.1)$.)

Example 6.26 *The charge-to-tap time (min) for carbon steel in one type of open hearth furnace is to be determined for each heat in a sample of size* n*. If the investigator believes that almost all times in the distribution are between 320 and 440, what sample size would be appropriate for estimating the true average time to within 5 minutes with a confidence level of* 95%*? (Answer: A reasonable value for* s *is* $(440 - 320)/4 = 30$*. Thus,*

$$n = \left(\frac{1.96 \times 30}{5}\right)^2 = 138.3\,,$$

that is, $n = 139$*.)*

7 Tests of Hypothesis (Chapter 8)

Definition 7.1 *A statistical hypothesis, or just hypothesis, is a claim or assertion either about the value of a single parameter, about the values of several parameters, or about the form of an entire probability distribution.*

Remark 7.2 *In any hypothesis-testing problem, there are two contradictory hypothesis under consideration. The objective is to decide, based on sample information, which of the two hypothesis is "correct". The problem will be formulated so that one of the claims is initially favored. The initially favored claim will not be rejected unless sample evidence contradicts it.*

Definition 7.3 *The null hypothesis, denoted by H_0, is the claim initially assumed to be true. The alternative hypothesis, denoted by H_a, is the assertion that is contradictory to H_0.*

Remark 7.4 *The null hypothesis will be rejected in favor of the alternative hypothesis only if sample evidence suggests that H_0 is "false". If the sample does not strongly contradict H_0, we will continue to believe in the plausibility of the null hypothesis. Roughly speaking, the two possible conclusions from a hypothesis-testing analysis are then: reject H_0, or fail to reject H_0.*

Definition 7.5 *A test of hypotheses is a method for using sample data to decide whether the null hypothesis should be rejected.*

Remark 7.6 *In our treatment of hypothesis testing, H_0 will generally be stated as an equality or inequality claim. If θ denotes the parameter of interest, the null hypothesis will have the form*

$$H_0 \,:\, \theta = \theta_0 \,,$$

(respectively, with $\geq$ or $\leq$) where θ_0 is a specified number called the null value of the parameter. The alternative to the null hypothesis will look like one of the following three assertions:

1. $H_a \,:\, \theta > \theta_0$.
2. $H_a \,:\, \theta < \theta_0$.
3. $H_a \,:\, \theta \neq \theta_0$.

Remark 7.7 *A test procedure is a rule, based on sample data, for deciding whether H_0 should be rejected. The key issue will be the following: suppose that H_0 is in fact true and we have a random sample. Then, how likely is it that a sample at least as contradictory to this hypothesis as our sample would result?*

Definition 7.8 *A test statistic is a function of the sample data used as a basis for deciding whether H_0 should be rejected. The P-value is the probability, calculated assuming that the null hypothesis is true, of obtaining a value of the test statistic at least as contradictory to H_0 as the value calculated from the available sample data.*

Remark 7.9 *A conclusion is reached in a hypothesis testing analysis by selecting a number* α*, called the* <u>*significance level*</u> *of the test, that is reasonably close to* 0*. Then,* H_0 *will be rejected in favor of* H_a *if the* P*-value is smaller or equal* α*. The smaller the significance level, the more protection is being given to the null hypothesis and the harder it is for that hypothesis to be rejected. Similarly, the smaller the* P*-value is, the stronger is the evidence against* H_0 *and in favor of* H_a*.*

Remark 7.10 *The* P*-value is not the probability that the null hypothesis is true or that it is false, nor is it the probability that an erroneous conclusion is reached.*

Example 7.11 *Assume a random sample of* $n = 51$ *batteries gave a sample mean zinc mass of* 2.06 *g and a sample standard deviation of* 0.141 *g. Does this data provide compelling evidence for concluding that the population mean zinc mass exceeds* 2.0 *g? Let's employ a significance level of* 0.01 *to reach a conclusion.*

If μ *denotes the true average zinc mass for such batteries, the relevant hypothesis are*

$$\begin{cases} H_0 \,:\, \mu = 2.0 \\ H_a \,:\, \mu > 2.0 \end{cases} .$$

The reasonably "large" sample size allows us to use the Central Limit Theorem, according to which the sample mean $\bar{X}$ *has approximately a normal distribution. Furthermore, the standardized variable*

$$Z = \frac{\bar{X} - \mu}{S/\sqrt{n}} \, ,$$

has approximately a standard normal distribution. The test statistic results from standardizing $\bar{X}$ *assuming that* H_0 *is true, i.e., the test statistic is*

$$Z = \frac{\bar{X} - 2}{S/\sqrt{n}} \, .$$

We now employ our sample for $n = 51$*,* $\bar{x} = 2.06$*, and* $s = 0.141$ *to obtain that* $z = 3.04$*. Any value of* $\bar{x}$ *larger than* 2.06 *is more contradictory to* H_0 *than* 2.06 *itself, and values of* $\bar{x}$ *that exceed* 2.06 *correspond to values of* z *that exceed* 3.04*. So, any* $z \geq 3.04$ *is at least as contradictory to* H_0*.*

Since the test statistic has approximately a standard normal distribution when H_0 *is true, we have that the* P*-value is more or less the probability that a standard normal random variable is bigger or equal* 3.04*, i.e.,* 0.0012*.*

Finally, because $0.0012 \leq 0.01 = \alpha$*, the null hypothesis should be rejected at the chosen significance level. It appears that true average zinc mass does indeed exceed* 2*.*

Definition 7.12 *A* <u>*type I error*</u> *consists of rejecting the null hypothesis* H_0 *when it is true. A* <u>*type II error*</u> *involves not rejecting* H_0 *when it is false.*

Proposition 7.13 *The test procedure that rejects H_0 if the P-value is smaller or equal α and otherwise does not reject H_0 has probability of type I error α. That is, the significance level employed in the test procedure is the probability of a type I error.*

Example 7.14 *An automobile model is known to sustain no visible damage 25% of the time in 10-mph crash tests. A modified bumped design has been proposed in an effort to increase this percentage. Let p denote the proportion of all 10-mph crashes with this new bumper that result in no visible damage. The hypothesis to be tested are:*

$$\begin{cases} H_0 \,:\, p = 0.25 \\ H_a \,:\, p > 0.25 \end{cases} .$$

Clearly, the null hypothesis H_0 means there is no improvement.

The test will be based on an experiment involving $n = 20$ independent crashes with prototypes of the new design. The natural test statistic here is X, the number of crashes with no visible damage. If H_0 is true, $E(X) = np_0 = 20 \times 0.25 = 5$. Intuition suggests that an observed value x much larger than this would provide strong evidence against H_0 and in support of H_a.

Consider using a significance level of $\alpha = 0.1$. The P-value is the probability of $X \geq x$ when X has a binomial distribution with $n = 20$ and $p = 0.25$. So,

$$P(X \geq x) = 1 - P(X < x) = 1 - P(X \leq x - 1) = 1 - B(x - 1; 20, 0.25) .$$

Computing some of these values we have $P(X \geq 7) = 0.214$, $P(X \geq 8) = 0.102$, and $P(X \geq 9) = 0.041$, hence, rejecting H_0 with the fixed significance level is equivalent to rejecting H_0 when $X \geq 9$ holds.

Example 7.15 (Monty Hall Problem (Part III)) *Make the experiment repeating the Monty Hall problem $n = 34$ times. Let X be the number of wins after switching. The wrong answer (as we know from the theory of probability) to the Monty Hall problem is that switching does not matter and so the probability of winning the car would be $1/3$. We want to verify experimentally that this wrong answer can be rejected with a small significance level α. The hypothesis to be tested are:*

$$\begin{cases} H_0 \,:\, p = 1/3 \\ H_a \,:\, p > 1/3 \end{cases} .$$

If H_0 is true we expect between 11 and 12 wins. An experimental value of wins x much larger than 12 would provide strong evidence against H_0 and in support of H_a.

The P-value is the probability of $X \geq x$ when X has a binomial distribution with $n = 34$ and $p = 1/3$. So,

$$P(X \geq x) = 1 - B(x - 1; 34, 1/3) .$$

Computing some values we have that $P(X \geq 16) = 0.07$ so, if x is 16 or more, the null hypothesis can be rejected at a significance level of $\alpha = 0.1$. If x is 17 or more, the null hypothesis H_0 can be rejected at a significance level $\alpha = 0.05$, since $P(X \geq 17) = 0.03$.

Example 7.16 *The drying time of a type of paint under specified test conditions is known to be normally distributed with mean value* 75 *minutes and standard deviation of* 9 *minutes. Chemists have proposed a new additive designed to decrease average drying time. It is believed that drying times with this additive will remain normally distributed with* $\sigma = 9$. *Because of the expense associated with the additive, evidence should strongly suggest an improvement in average drying time before such a conclusion is adopted.*

Let μ *denote the true average drying time when the additive is used. The appropriate hypotheses are:*

$$\begin{cases} H_0 \,:\, \mu = 75 \\ H_a \,:\, \mu < 75 \end{cases} .$$

Only if H_0 *can be rejected will the additive be declared successful and used.*

Experimental data is to consist of drying times from $n = 25$ *test specimens. Let* $X_1,\ldots,X_{25}$ *denote the* 25 *drying times. The sample mean drying time* $\bar{X}$ *then has a normal distribution with expected value* $\mu_{\bar{X}} = \mu$ *and standard deviation* $\sigma_{\bar{X}} = \sigma/\sqrt{n} = 1.8$. *When* H_0 *is true, we expect* $\bar{X}$ *to be* 75*; a sample mean much smaller than this would be contradictory to* H_0 *and supportive of* H_a.

Our test statistic here will be $\bar{X}$ *standardized normal assuming* H_0 *is true:*

$$Z = \frac{\bar{X} - 75}{1.8} .$$

The sampling distribution of $\bar{X}$ *is normal because the population distribution is normal, which implies that* Z *has a standard normal distribution when* H_0 *is true.*

Consider carrying out the test using a significance level of $\alpha = 0.01$. *For a given value* $\bar{x}$ *of the sample mean and corresponding calculated value* z*, the form of the alternative hypothesis implies that values more contradictory to* H_0 *than this are values less than* $\bar{x}$ *and, correspondingly, values of the test statistic that are less than* z. *Thus, the* P*-value is the probability of* $Z \leq z$ *when* H_0 *is true. The* P*-value will equal* $\alpha = 0.01$ *when* z *captures lower-tail area of* 0.01 *under the* z *curve, which happens precisely at* $z = -2.33$.

Example 7.17 *A manufacturer of sprinkler systems used for fire protection in office buildings claims that the true average system-activation temperature is* 130. *A sample of* $n = 9$ *systems, when tested, yields a sample average activation temperature of* 131.08. *If the distribution of activation times is normal with standard deviation* 1.5*, does the data contradict the manufacturer's claim at significance level* $\alpha = 0.01$*?*

1. *Parameter of interest:* μ *is the true average activation temperature.*
2. *Null hypothesis:* $H_0 \,:\, \mu = 130$ *(null value is* $\mu_0 = 130$*).*
3. *Alternative hypothesis:* $H_a \,:\, \mu \neq 130$ *(a departure from the claimed value in either direction is of concern).*

4. *Test statistic:*

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{X} - 130}{1.5/\sqrt{n}}\,.$$

5. *Substituting* $n = 9$ *and* $\bar{x} = 131.08$, *we get* $z = 2.16$. *That is the observed sample mean is a bit more than* 2 *standard deviations above what would have been expected were* H_0 *true.*

6. *The inequality in* H_a *implies that the test is two-tailed, so the* P*-value results from doubling the captured tail area, i.e.,* 0.0308.

7. *Since the* P*-value is bigger than* α, H_0 *cannot be rejected at that significance level. The data does not give strong support to the claim that the true average differs from the design value of* 130.

Example 7.18 *Carbon nanofibers have potential application as heat-management materials, for composite reinforcement, and as components for nanoelectronics and photonics. The accompanying data on failure stress of a fiber specimens is:*

$$300\,,\quad 312\,,\quad 327\,,\quad 368\,,\quad 400\,,\quad 425\,,\quad 470\,,\quad 556\,,\quad 573\,,\quad 575\,,$$
$$580\,,\quad 589\,,\quad 626\,,\quad 637\,,\quad 690\,,\quad 715\,,\quad 757\,,\quad 891\,,\quad 900\,.$$

We then have $n = 19$, $\bar{x} = 562.68$, $s = 180.874$, $s/\sqrt{n} = 41.495$. *Does the data provide compelling evidence for concluding that true average failure stress exceeds* 500*?*

Assuming a normal distribution, let's carry out a test of the relevant hypotheses using a significance level of $\alpha = 0.05$.

1. *The parameter of interest is* μ, *the true average failure stress.*

2. *The null hypothesis is* $H_0 : \mu = 500$.

3. *The appropriate alternative hypothesis is* $H_a : \mu > 500$ *(so we will believe that true average failure stress exceeds* 500 *only if the null hypothesis can be rejected).*

4. *The one-sample* t *test statistic is*

$$T = \frac{\bar{X} - 500}{S/\sqrt{n}}\,.$$

   Its value t *for the given data results from replacing* $\bar{X}$ *by* $\bar{x}$ *and* S *by* s.

5. *The test-statistic value is* $t = 1.51$.

6. *The test is based on* $19 - 1$ *degrees of freedom. So, the* P*-value is more or less* 0.075.

7. *Since the* P*-value is bigger than* α, *there is not enough evidence to justify rejecting the null hypothesis at that significance level.*

Example 7.19 *Many deleterious effects of smoking on health have been well documented. An investigation into whether time perception, an indicator of a person's ability to concentrate, is impaired during nicotine withdrawal was described in one of these investigations. After a 24-hour smoking abstinence, each of* 20 *smokers was asked to estimate how much time had elapsed during a* 45*-second period. Here is the data:*

$$\begin{array}{cccccccccc} 69, & 65, & 72, & 73, & 59, & 55, & 39, & 52, & 67, & 57, \\ 56, & 50, & 70, & 47, & 56, & 45, & 70, & 64, & 67, & 53. \end{array}$$

Assuming the normal distribution of the population, let's carry out a test of hypotheses at significance level $\alpha = 0.05$ *to decide whether true average perceived elapsed time differs from the known time* 45*.*

1. *Denote by* μ *the true average perceived elapsed time for all smokers exposed to the described experimental regimen.*
2. *The null hypothesis is* $H_0 : \mu = 45$*.*
3. *The alternative hypothesis is* $H_a : \mu \neq 45$*.*
4. *The one-sample* t *test statistic is*
$$T = \frac{\bar{X} - 45}{S/\sqrt{n}}\,.$$
5. *Using* $\bar{x}$*,* s *and* n*, its value is* $t = 6.5$*.*
6. *The* P*-value value for a two-tailed test is twice the area under the* 19 *degrees of freedom* t *curve, which is approximately* 0*.*
7. *A* P*-value as small as what we obtained argues very strongly for rejection of* H_0 *at any reasonable significance level.*

8 Statistical Inference (Chapter 9)

Remark 8.1 *We have already studied confidence intervals and hypothesis-testing procedures for a single sample. Here, we extend these methods to situations involving two different population distributions.*

8.1 z Tests

Remark 8.2 *The basic assumptions are the followings:*

1. $X_1,\ldots,X_m$ *is a random sample from a normal distribution with mean* μ_1 *and variance* σ_1^2.
2. $Y_1,\ldots,Y_n$ *is a random sample from a normal distribution with mean* μ_2 *and variance* σ_2^2.
3. *The* X *and* Y *samples are independent of one another.*

Proposition 8.3 *The expected value of* $\bar{X}-\bar{Y}$ *is* $\mu_1-\mu_2$*, so* $\bar{X}-\bar{Y}$ *is an unbiased estimator of* $\mu_1-\mu_2$*. The standard deviation of* $\bar{X}-\bar{Y}$ *is*

$$\sigma_{\bar{X}-\bar{Y}}=\sqrt{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}}.$$

Remark 8.4 *Since the population distributions are normal, both* $\bar{X}$ *and* $\bar{Y}$ *have normal distributions. Furthermore, independence of the two samples implies that the two sample means are independent of one another. Thus, the difference* $\bar{X}-\bar{Y}$ *is normally distributed, with expected value* $\mu_1-\mu_2$ *and standard deviation* $\sigma_{\bar{X}-\bar{Y}}$.

Example 8.5 *Analysis of a random sample consisting of* $m = 20$ *specimens of cold-rolled steel to determine yield strengths resulted in a sample average strength of* $\bar{x} = 29.8$*. A second random sample of* $n = 25$ *two-sided galvanized steel specimens gave a sample average strength of* $\bar{y} = 34.7$*. Assuming that the two yield-strength distributions are normal with* $\sigma_1 = 4$ *and* $\sigma_2 = 5$*, does the data indicate that the corresponding true average yield strengths* μ_1 *and* μ_2 *are different?*

Let's carry out a test at significance level $\alpha = 0.1$.

1. *The parameter of interest is* $\mu_1-\mu_2$*, the difference between the true average strengths for the two types of steel.*
2. *The null hypothesis is* $H_0 : \mu_1-\mu_2 = 0$.
3. *The alternative hypothesis is* $H_a : \mu_1-\mu_2 \neq 0$.
4. *The test statistic value is then*

$$z=\frac{\bar{x}-\bar{y}-\mu_1+\mu_2}{\sqrt{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}}}.$$

5. *Substituting* $m = 20$, $\bar{x} = 29.8$, $\sigma_1^2 = 16$, $n = 25$, $\bar{y} = 34.7$, *and* $\sigma_2^2 = 25$, *yields* $z = -3.66$. *That is, the observed value of* $\bar{x} - \bar{y}$ *is more than 3 standard deviations below what would be expected were* H_0 *true.*

6. *The* P*-value is then approximately* 0.

7. *Consequently, since the* P*-value is smaller than* α, H_0 *is rejected at level* 0.01 *in favor of the conclusion that* $\mu_1 \neq \mu_2$.

Example 8.6 *Of* 215 *male physicians who were Harvard graduates and died between November 1974 and October 1977, the* 125 *in full-time practice lived an average of* 48.9 *years beyond graduation, whereas the* 90 *with academic affiliations lived an average of* 43.2 *years beyond graduation. Does the data suggest that the mean lifetime after graduation for doctors in full-time practice exceeds the mean lifetime for those who have academic affiliation?*

Let μ_1 *denote the true average number of years lived beyond graduation for physicians in full-time practice, and let* μ_2 *denote the same quantity for physicians with academic affiliations. Assume the* 125 *and* 90 *physicians to be random samples from populations one and two, respectively. Assume that* $\sigma_1 = 14.6$ *and* $\sigma_2 = 14.4$. *The hypotheses are* $H_0 : \mu_1 - \mu_2 = 0$ *versus* $H_a : \mu_1 - \mu_2 > 0$. *The computed value of the test statistic is*

$$z = \frac{48.9 - 43.2}{\sqrt{\frac{(14.6)^2}{125} + \frac{(14.4)^2}{90}}} = 2.85\,.$$

The P*-value is* 0.0022. *At significance level* $\alpha = 0.01$, H_0 *is rejected in favor of the conclusion that* $\mu_1 > \mu_2$.

Remark 8.7 *The assumptions of normal population distributions and known values of* σ_1 *and* σ_2 *are, fortunately, unnecessary when both sample sizes are sufficiently large. In this case, the Central Limit Theorem guarantees that* $\bar{X} - \bar{Y}$ *has approximately a normal distribution regardless of the underlying population distributions.*

Example 8.8 *What impact does fast-food consumption have on various dietary and health characteristics? The following data is about the daily calorie intake both for a sample of teens who said they did not typically eat fast food and another sample of teens who said they did usually eat fast food:*
No eat fast food: $m = 663$, $\bar{x} = 2258$ *and* $s_1 = 1519$.
Yes eat fast food: $n = 413$, $\bar{y} = 2637$ *and* $s_2 = 1138$.
Does this data provide strong evidence for concluding that true average calorie intake for teens who typically eat fast food exceeds by more than 200 *calories per day the true average intake for those who do not typically eat fast food? Let's carry out a test of hypotheses at a significance level of* $\alpha = 0.05$.

The parameter of interest is $\mu_1 - \mu_2$, *where* μ_1 *is the true average calorie intake for teens who do not typically eat fast food and* μ_2 *is the true average intake for teens who do typically* *eat fast food. The hypotheses of interest are:*

$$\begin{cases} H_0 \,:\, \mu_1 - \mu_2 = -200 \\ H_a \,:\, \mu_1 - \mu_2 < -200 \end{cases}.$$

The test statistic value is

$$z = \frac{\bar{x} - \bar{y} + 200}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} = -2.2\,.$$

The inequality in H_a implies that the P-value is 0.0139*. Since this value is smaller than α, the null hypothesis is rejected.*

Definition 8.9 *Provided that m and n are both large, a* <u>*confidence interval*</u> *for $\mu_1 - \mu_2$ with a confidence level of approximately* $100(1-\alpha)\%$ *is*

$$\left(\bar{x} - \bar{y} - z_{\alpha/2}\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}, \bar{x} - \bar{y} + z_{\alpha/2}\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}\right).$$

8.2 t Tests

Remark 8.10 *The basic assumptions are the followings:*

1. *X_1,...,X_m is a random sample from a normal distribution.*
2. *Y_1,...,Y_n is a random sample from a normal distribution.*

Theorem 8.11 *The standardized variable*

$$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}}\,,$$

where μ_1 and μ_2 are the mean values of X and Y, respectively, and S_1 and S_2 are the corresponding standard deviations, has approximately a t distribution with ν degrees of freedom. The number ν can be estimated from the data by

$$\nu = \frac{\left(\frac{s_1^2}{m} + \frac{s_2^2}{n}\right)^2}{\frac{(s_1^2/m)^2}{m-1} + \frac{(s_2^2/n)^2}{n-1}}\,,$$

and rounding it down to the nearest integer.

Definition 8.12 *The* <u>*two-sample t confidence interval*</u> *for $\mu_1 - \mu_2$ with confidence level* $100(1-\alpha)\%$ *is*

$$\left(\bar{x} - \bar{y} - t_{\alpha/2,\nu}\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}, \bar{x} - \bar{y} + t_{\alpha/2,\nu}\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}\right).$$

Example 8.13 *The void volume within a textile fabric affects comfort, flammability, and insulation properties. Permeability of a fabric refers to the accessibility of void space to the flow of a gas or liquid. The following is summary information on air permeability for a number of different fabric types:*
Cotton: $m = 10$, $\bar{x} = 51.71$ *and* $s_1 = 0.79$.
Triacetate: $n = 10$, $\bar{y} = 136.14$ *and* $s_2 = 3.59$.
Assuming that the porosity distributions for both types of fabric are normal, let's calculate a confidence interval for the difference between true average porosity for the cotton fabric and that for the triacetate fabric, using a 95% *confidence level. (Answer: The degrees of freedom can be approximated by:* $\nu = 9.87$, *so we will consider* $\nu = 9$. *For this value,* $t_{0.025,9} = 2.262$ *and the resulting interval is:* $(-87.06, -81, 8)$.*)*

8.3 Analysis of Paired Data

Remark 8.14 *Up to now we have made inference utilizing the results of random samples of different number of individuals from distributions. In contrast there are a number of experimental situations in which there is only one set of* n *individuals, making two observations on each one results in a natural pairing of values.*

Remark 8.15 *The basic assumptions are the followings:*

1. *The data consists of* n *independently selected pairs* $(X_1, Y_1), ..., (X_n, Y_n)$, *with* $E(X_i) = \mu_1$ *and* $E(Y_i) = \mu_2$.
2. *Denote by* $D_i = X_i - Y_i$ *the differences within the pairs. These values are normally distributed with mean value* μ_D *and variance* σ_D^2 *(this is usually a consequence of* $X_1, ..., X_n$ *and* $Y_1, ..., Y_n$ *being normally distributed).*

Remark 8.16 *Since the differences* D_i *constitute a normal random sample with mean* μ_D, *hypothesis about* μ_D *can be tested using a one-sample* t *test.*

Example 8.17 *Musculoskeletal neck-and-shoulder disorders are all too common among office staff who perform repetitive tasks using visual display units. A study was reported to determine whether more varied work conditions would have any impact on arm movement. The accompanying data was obtained from a sample* $n = 16$ *subjects. Each observation is the amount of time, expressed as a proportion of total time observed, during which arm elevation was below* 30 *The two measurements from each subject were obtained* 18 *months apart. During this period, work conditions were changed, and subjects were allowed to engage in a wider variety of work tasks. Does the data suggest that true average time during which elevation is below* 30 *differs after the change from what it was before the change?*

These are the differences:

$$81-78=3\,,\quad 87-91=-4\,,\quad 86-78=8\,,\quad 82-78=4\,,\quad 90-84=6\,,$$
$$86-67=19\,,\quad 96-92=4\,,\quad 73-70=3\,,\quad 74-58=16\,,\quad 75-62=13\,,$$
$$72-70=2\,,\quad 80-58=22\,,\quad 66-66=0\,,\quad 72-60=12\,,\quad 56-65=-9\,,$$
$$82-73=9\,.$$

Assume these differences are normally distributed and test the appropriate hypothesis.

1. *Let μ_D denote the true average difference between elevation time before the change in work conditions and time after the change.*
2. *The null hypothesis will be $H_0 : \mu_D = 0$ (there is no difference between true average time before the change and true average time after the change).*
3. *The alternative hypothesis is $H_a : \mu_D \neq 0$.*
4. *We will use the test statistic value*
$$t = \frac{\bar{d}-0}{s_D/\sqrt{n}} = 3.3\,.$$
5. *The P-value is approximately* 0.004.
6. *When $\alpha > 0.004$, the null hypothesis can be rejected.*

8.4 Inferences Concerning Two Population Variances

Definition 8.18 *The F probability distribution is a distribution which has two parameters $\nu_1 > 0$ and $\nu_2 > 0$, called the number of numerator degrees of freedom and number of denominator degrees of freedom. (We will omit the explicit expression of the associated density distribution, since it is rather complicated).*

Proposition 8.19 *If X_1 and X_2 are two independent χ^2 random variables with ν_1 and ν_2 degrees of freedom, respectively, then the random variable*

$$F = \frac{X_1/\nu_1}{X_2/\nu_2}\,,$$

has an F distribution.

Theorem 8.20 *Let $X_1,\ldots,X_m$ be a random sample from a normal distribution with variance σ_1^2 and $Y_1,\ldots,Y_n$ another random sample independent of X from a normal distribution with variance σ_2^2. Assume that S_1^2 and S_2^2 denote the two sample variances. Then the random variable*

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

has an F distribution with $\nu_1 = m-1$ and $\nu_2 = n-1$.

Example 8.21 *A random sample of* 200 *vehicles traveling on gravel roads in a county with posted speed limit of* 35 *mph on such roads resulted in a sample mean speed of* 37.5 *mph and a sample standard deviation of* 8.6 *mph, whereas another random sample of* 200 *vehicles in a county with posted speed limit of* 55 *mph resulted in a sample mean and sample standard deviation of* 35.8 *mph and* 9.2 *mph, respectively.*

Let's carry out a test at significance level $\alpha = 0.1$ *to decide whether the two population distribution variances are identical.*

1. *Denote by* σ_1^2 *the variance of the speed distribution on the* 35 *mph roads, and by* σ_2^2 *the variance of the speed distribution on* 55 *mph roads.*

2. *The null hypothesis is* $H_0 : \sigma_1^2 = \sigma_2^2$.

3. *The alternative hypothesis is* $H_a : \sigma_1^2 \neq \sigma_2^2$.

4. *The test statistic value is* $f = s_1^2/s_2^2 = 0.87$.

5. *We have a* F *distribution with* 199 *numerator degrees of freedom and* 199 *denominator degrees of freedom. Then,* $F_{0.1,199,199} = 0.83$. *The* P*-value is approximately* 0.342.

6. *The* P*-value clearly exceeds the significance level. Therefore, the null hypothesis cannot be rejected.*

Review Problems

1. Assume that the helium porosity (in percentage) of coal samples taken from any particular seam is normally distributed with true standard deviation 0.75.
   (a) Compute a 95% confidence interval for the true average porosity of a certain seam if the average porosity for 20 specimens from the seam was 4.85. (Answer: $(4.52, 5.18)$.)
   (b) How large a sample size is necessary if the width of the 95% interval is to be 0.4? (Answer: 55.)

2. For a sample of 50 kitchens with gas cooking appliances monitored during a one-week period, the sample mean CO_2 level (ppm) was 654.16, and the sample standard deviation was 164.43.
   (a) Calculate a 95% confidence interval for true average CO_2 level in the population of all homes from which the sample was selected. (Answer: $(608.58, 699.74)$.)
   (b) How large a sample size is necessary if the width of the 95% interval is to be 50? (Answer: 167.)

3. The following observations on degree of polymerization for paper specimens for which viscosity times concentration fell in a certain middle range are given:

$$418\,,\quad 421\,,\quad 421\,,\quad 422\,,\quad 425\,,\quad 427\,,\quad 432\,,\quad 434\,,\quad 437\,,$$
$$439\,,\quad 446\,,\quad 447\,,\quad 448\,,\quad 453\,,\quad 454\,,\quad 463\,,\quad 465\,.$$

   Assuming they were selected from a normal distribution, calculate a 95% confidence interval for the true average degree of polymerization. (Hint: $t_{0.025,16} = 2.12$. Answer: $(430.5, 446.1)$.)

4. A sample of 50 lenses used in eyeglasses yields a sample mean thickness of $0.34mm$. The desired true average thickness of such lenses is $3.2mm$. Does the data strongly suggest that the true average thickness of such lenses is something other than what is desired? Test using a significance level $\alpha = 0.05$. (Hint: $P(Z \leq 3.12) = 0.9991$.)

5. The following summary data on daily caffeine consumption for a sample of adults is given: $n = 47$, $\bar{x} = 215mg$ and $s = 235mg$. Suppose that it has previously been believed that mean consumption was at most $200mg$. Does the given data contradict this prior belief? Use a significance level of $\alpha = 0.1$. (Hint: $P(Z \leq 0.44) = 0.67$.)

6. The true average breaking strength of ceramic insulators of a certain type is supposed to be at least $10psi$. They will be used for a particular application unless sample data indicates conclusively that this specification has not been met. A test of hypotheses using a significance level $\alpha = 0.01$ is to be based on a random sample of 10 insulators which gives a sample mean of $9psi$. Assume that the breaking-strength distribution is normal and:

(a) The true standard deviation is 0.8. (Hint: $P(Z \leq 1.25) = 0.8944$.)

(b) The sample standard deviation is 0.8. (Hint: $P(Z \leq 1.25) = 0.875$.)

7. Wire electrical-discharge machining (WEDM) is a process used to manufacture conductive hard metal components. It uses a continuously moving wire that serves as an electrode. Coating on the wire electrode allows for cooling of the wire electrode core and provides an improved cutting performance. The following sample observations on total coating layer thickness (in μm) of eight wire electrodes used for WEDM is given:

$$21\,,\quad 16\,,\quad 29\,,\quad 35\,,\quad 42\,,\quad 24\,,\quad 24\,,\quad 25\,.$$

   Calculate a 99% confidence interval for the standard deviation of the coating layer thickness distribution. (Answer: $(4.82, 21.85)$.)

8. The following data on time to repair (min) a rail break in high rail on a curved track of a certain railway line is presented:

$$159\,,\quad 120\,,\quad 480\,,\quad 149\,,\quad 270\,,\quad 547\,,\quad 340\,,\quad 43\,,\quad 228\,,\quad 202\,,\quad 240\,,\quad 218\,.$$

   Assume that the population distribution of repair time is normal. Is there compelling evidence for concluding that true average repair time exceeds 200 min? Carry out a test of hypotheses using a significance level of $\alpha = 0.05$. (Hint: $P(T_{11} \leq 1.2) = 0.872$.)

9. The sample average unrestrained compressive strength for 45 specimens of a particular type of brick was computed to be $3107psi$, and the sample standard deviation was 188. Does the data strongly indicate that the true average unrestrained compressive strength is less than the design value 3200? Test using $\alpha = 0.001$. (Hint: $P(Z \leq 3.32) = 0.9995$.)

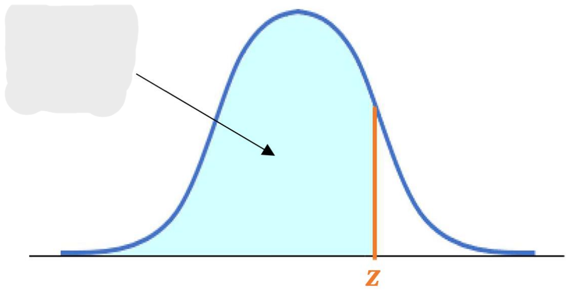
10. The American Academy of Pediatrics recommends a vitamin D level of at least 20ng/ml for infants. A sample of 102 preterm infants judged to be of appropriate weight for their gestational age shows a sample mean vitamin D level at 2 weeks of 21 with sample standard deviation of 11. Does this provide convincing evidence that the population mean vitamin D level for such infants exceed 20? Test the relevant hypotheses using a significance level of $\alpha = 0.1$. (Hint: $P(Z \leq 0.92) = 0.8212$.)

11. Is there any systematic tendency for part-time college faculty to hold their students to different standards than do full-time faculty? An article reported that for a sample of 125 courses taught by full-time faculty, the mean course GPA was 2.7186 and the standard deviation was 0.63342, whereas for a sample of 88 courses taught by part-timers, the mean and standard deviation were 2.8639 and 0.49241, respectively. Does it appear that true average course GPA for part-time faculty differs from that for faculty teaching full-time? Test the appropriate hypotheses at significance level $\alpha = 0.01$. (Hint: $P(Z \leq 1.88) = 0.9699$.)

12. Suppose μ_1 and μ_2 are true mean stopping distances at $50mph$ for cars of a certain type equipped with two different types of breaking systems. Assuming that the populations

are normally distributed, test at significance level $\alpha = 0.01$ whether μ_2 exceeds by more than 10 units μ_1. Use the following data: $m = 6$, $\bar{x} = 115.7$, $s_1 = 5.03$, $n = 6$, $\bar{y} = 129.3$ and $s_2 = 5.38$. (Hint: $P(T_9 \leq 1.2) = 0.87$.)

Appendix A. Table of the Standard Normal Distribution



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5754
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7258	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7518	0.7549
0.7	0.7580	0.7612	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7996	0.8023	0.8051	0.8079	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9430	0.9441
1.6	0.9452	0.9463	0.9474	0.9485	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9700	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9762	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9983	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	0.9998

Appendix B. Table of the Student's t Distribution

cum. prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tail	**0.50**	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
two-tails	**1.00**	**0.50**	**0.40**	**0.30**	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.002**	**0.001**
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										