Variational Theory of Membranes

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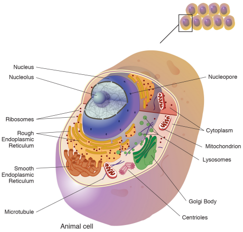
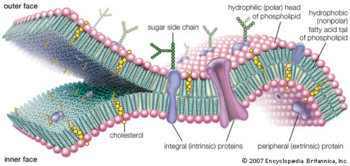
Modeling Biological Membranes





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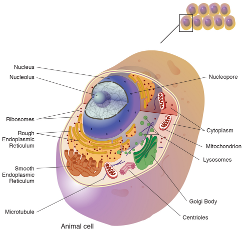

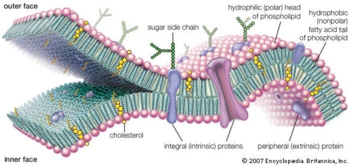

W. Helfrich (1973) suggested to study the critical points of

$$\mathcal{H}[\Sigma] := \int_\Sigma \left(a\left[H + c_o\right]^2 + bK \right) d\Sigma \,,$$

to model biological membranes.

The Helfrich Energy

Let Σ be a compact (with or without boundary) surface. For an embedding $X : \Sigma \to \mathbb{R}^3$ the Helfrich energy is given by

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where the energy parameters are:

- The bending rigidity: $a > 0$.
- The spontaneous curvature: $c_o \in \mathbb{R}$.
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Gauss-Bonnet Theorem

The total Gaussian curvature term only affects the boundary.

Euler-Lagrange Equation

The Euler-Lagrange equation associated to $\mathcal{H}$ is

$$\Delta(H + c_o) + 2\,(H + c_o)\,(H\,[H - c_o] - K) = 0\,,$$

a fourth order nonlinear elliptic PDE.

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1. Constant Mean Curvature Surfaces with $H \equiv -c_o$.

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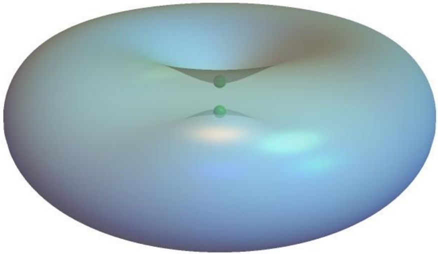
(Far from the axis of rotation.)

Circular Biconcave Discoids



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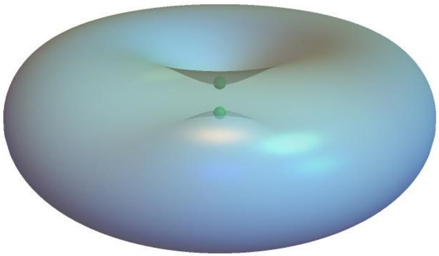

Proposition (López, Palmer & P., Preprint)

Let $\psi \in \mathcal{C}_o^\infty(\Sigma)$ and consider normal variations $\delta X = \psi\nu$, then

$$\delta\mathcal{H}[\Sigma] = 8\pi c_o \psi_{|_{r=0}} .$$

Axially Symmetric Solutions

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Theorem (Palmer & P., 2022)

An axially symmetric disc critical for $\mathcal{H}$ must be:

(I) A planar disc ($H \equiv -c_o = 0$).

(II) A spherical cap ($H \equiv -c_o \neq 0$).

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- The surface must be a topological disc. Annular domains in circular biconcave discoids are critical for $\mathcal{H}$.

Reduced Membrane Equation

The reduced membrane equation is the Euler-Lagrange equation for

$$\mathcal{G}[\Sigma] := \int_{\Sigma} \frac{1}{z^2}\, d\Sigma - 2c_o \int_{\Omega} \frac{1}{z^2}\, d\Sigma\,.$$

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Solutions can be viewed as:

- Capillary surfaces with constant gravity in $\mathbb{H}^3$.
- Weighted CMC surfaces for the density $\phi = -2\log(z)$.
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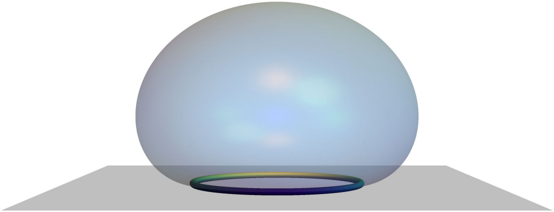
- The right cylinders over elastic curves satisfy the reduced membrane equation.

Symmetry Breaking Bifurcation



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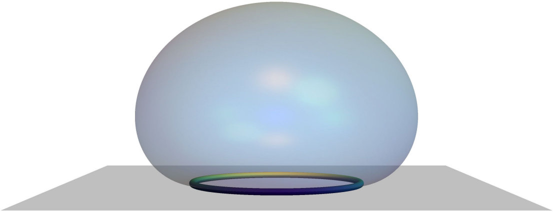

Theorem **(Palmer & P., 2024)**

Above surface Σ_0 is embedded in a one parameter family of axially symmetric solutions of the reduced membrane equation (parameterized by c_o) which all share the same boundary circle.

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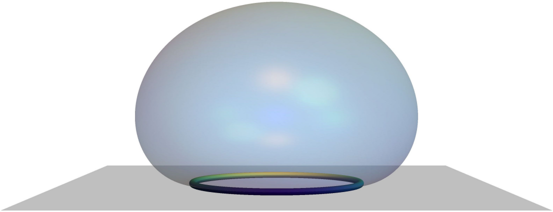
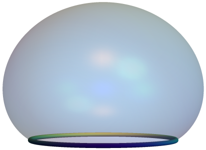


Theorem (Palmer & P., 2024)

Above surface Σ_0 is embedded in a one parameter family of axially symmetric solutions of the reduced membrane equation (parameterized by c_o) which all share the same boundary circle. Precisely, at Σ_0, a non-axially symmetric branch bifurcates.

Axially Symmetric Family



Theorem (Palmer & P., Preprint)

Subdomains of Σ_0 are stable and superdomains of Σ_0 are unstable for the functional $\mathcal{G}$.

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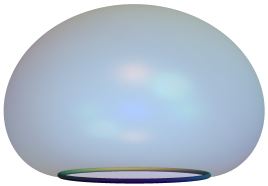

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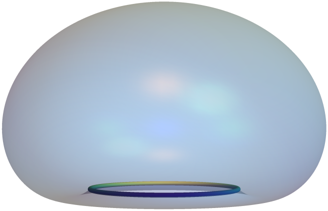

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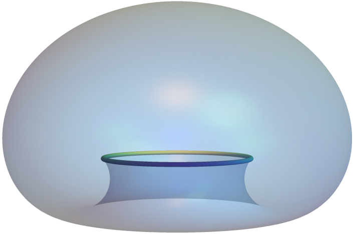

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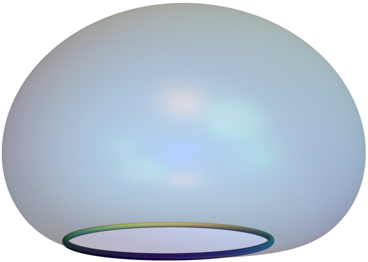

Conjecture

It is a subcritical pitchfork bifurcation.

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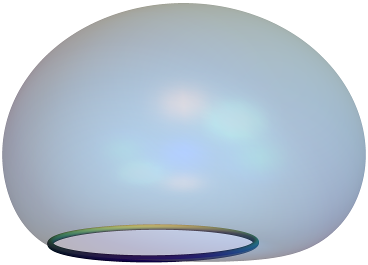

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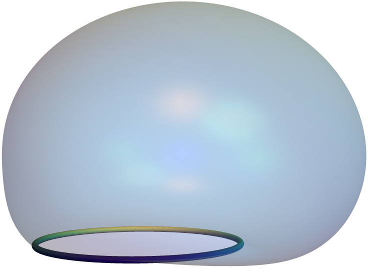

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Modified (Conformal) Gauss Map

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For a real constant c_o we define the map $Y^{c_o} : \Sigma \to \mathbb{S}^4_1 \subset \mathbb{E}^5_1$ by

$$Y^{c_o} := (H + c_o)\,\underline{X} + (\nu, q, q),$$

where $q := X \cdot \nu$ is the support function and

$$\underline{X} := \left(X, \frac{X^2 - 1}{2}, \frac{X^2 + 1}{2} \right).$$

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Theorem (Palmer & P., 2022)

The immersion $X : \Sigma \to \mathbb{R}^3$ is critical for the Helfrich energy $\mathcal{H}$ with respect to compactly supported variations if and only if

$$\Delta Y^{c_o} + \|dY^{c_o}\|^2 Y^{c_o} = 2c_o(0, 0, 0, 1, 1)^T.$$

(The map Y^{c_o} fails to be an immersion where $H^2 - K = c_o^2$.)

Special Solutions

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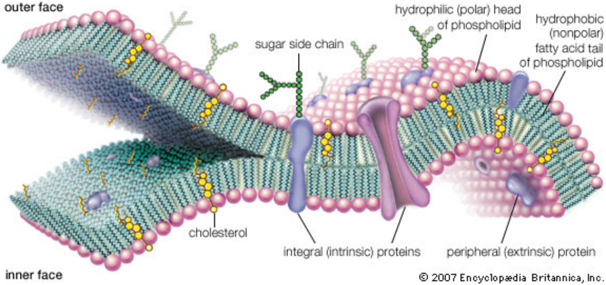
3. Case $\omega := (0,0,1,0,0)$ is a spacelike vector. Then,

$$H + c_o = -\frac{\nu_3}{z} \, .$$

(The Reduced Membrane Equation.)

Boundary Problems



The Euler-Helfrich Problem

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$$E[\Sigma] := \int_{\Sigma} \left(a\,[H + c_o]^2 + bK \right) d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \beta\right) ds\,,$$

where $\alpha > 0$ and $\beta > 0$.

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Boundary Conditions

The Euler-Lagrange equations on the boundary $\partial\Sigma$ are:

$$\begin{aligned} a\left(H + c_o\right) + b\kappa_n &= 0\,, \\ J' \cdot \nu - a\partial_n H + b\tau_g' &= 0\,, \\ J' \cdot n + a\left(H + c_o\right)^2 + bK &= 0\,, \end{aligned}$$

where J is a vector field along $\partial\Sigma$ defined by

$$J := 2\alpha T'' + \left(3\alpha\kappa^2 - \beta\right) T\,.$$

Ground State Equilibria

Assume $H + c_o \equiv 0$ holds on Σ. Then, the Euler-Lagrange equations reduce to

$$\begin{aligned} b\kappa_n &= 0\,, && \text{on } \partial\Sigma\,, \\ J' \cdot \nu + b\tau_g' &= 0\,, && \text{on } \partial\Sigma\,, \\ J' \cdot n - b\tau_g^2 &= 0\,, && \text{on } \partial\Sigma\,. \end{aligned}$$

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Proposition **(Palmer & P., 2021)**

Let $X : \Sigma \to \mathbb{R}^3$ be an equilibrium with $H + c_o \equiv 0$. Then, each boundary component C is a simple and closed critical curve for

$$F[C] \equiv F_{\mu,\lambda}[C] := \int_C \left([\kappa + \mu]^2 + \lambda \right) ds\,,$$

where $\mu := \pm b/(2\alpha)$ and $\lambda := \beta/\alpha - \mu^2$.

Results of Topological Discs

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Theorem (Palmer & P., 2021)

Let $X : \Sigma \to \mathbb{R}^3$ be a CMC $H = -c_o$ disc type surface critical for E. Then:

1. Case $b = 0$. The boundary is either a circle of radius $\sqrt{\alpha/\beta}$ or a simple closed elastic curve representing a torus knot of type $G(q, 1)$ for $q > 2$.
2. Case $b \neq 0$. The surface is a planar disc bounded by a circle of radius $\sqrt{\alpha/\beta}$ and $c_o = 0$.

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Idea of the proof:

- Elastic curves are torus knots $G(q, p)$ with $2p < q$ and the surface is a Seifert surface.

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Idea of the proof:

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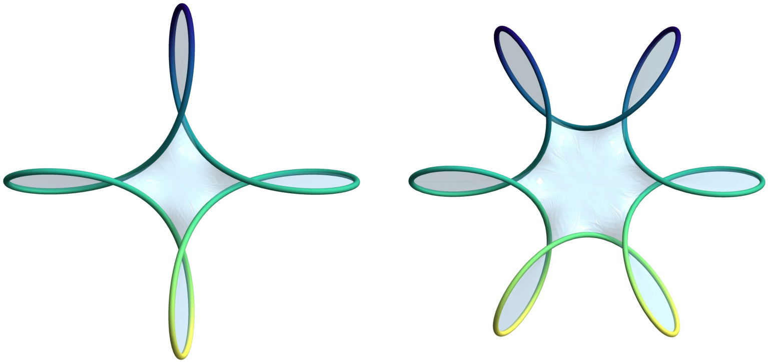
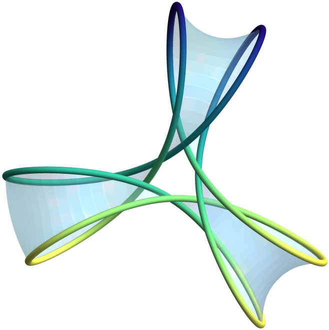
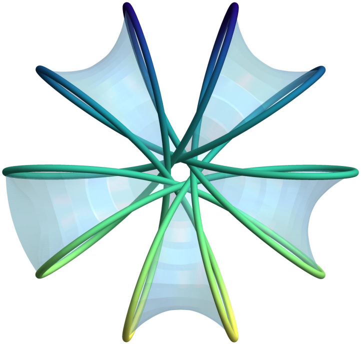
- Nitsche's argument involving the Hopf differential.

Minimal Discs Spanned by Elastic Curves



Minimal Annuli Spanned by Elastic Curves

Absolute Minimizers

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Theorem (Palmer & P., 2021)

The Euler-Helfrich energy E is bounded below if and only if

$$\underline{E} := 2\sqrt{\alpha\,\beta} - |b| \geq 0\,.$$

For the lower bound to be attained, the surface must have $H \equiv -c_o$ and the boundary must be composed by circles of radius $\sqrt{\alpha/\beta}$. In addition, either $b = 0$ or $\kappa_n \equiv 0$ must hold along the boundary.

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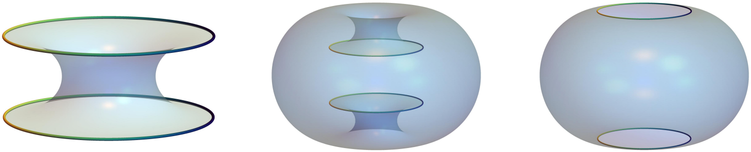
- In the case of a topological annulus, the lower bound can always be attained or approached.

Nodoidal Domains



Topological Discs

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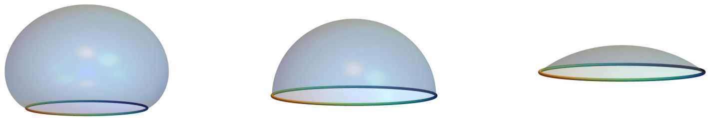

Conjecture

If $\underline{E} \geq 0$ holds, the infimum of the Euler-Helfrich energy E is attained by an axially symmetric surface with non-constant mean curvature. (Reduced Membrane Equation.)

Second Variation Formula

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Theorem (Palmer & P., Preprint)

Let $X : \Sigma \to \mathbb{R}^3$ be an immersion critical for the Helfrich energy $\mathcal{H}$ satisfying the reduced membrane equation. Then, for every $f \in \mathcal{C}_o^\infty(\Sigma)$,

$$\delta^2\mathcal{H}[\Sigma] = \int_\Sigma f\, F[f]\, d\Sigma + \frac{1}{2}\int_{\partial\Sigma} L[f]\,\partial_n f\, ds\,,$$

where

$$F[f] := \frac{1}{2}\left(P^* + \frac{2}{z^2}\right) \circ P[f]\,.$$

(Here, P is the operator arising as twice the variation of the quantity $H + \nu_3/z$, and P^* is its adjoint operator.)

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- Compute the second variation through the flux formula.

THE END

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- B. Palmer and A. Pámpano, Symmetry Breaking Bifurcation of Membranes with Boundary, *Nonlinear Anal.* **238** (2024), 113393.
- B. Palmer and A. Pámpano, Stability of Membranes, *Preprint*, arXiv: 2401.05285 [math.DG].

Thank You!