Some Geometric Variational Open Problems

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The Principle of Least Action

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- Often attributed to P. L. Maupertuis (1744-1746).
- Already known to G. Leibniz (1705) and L. Euler (1744).

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- Although the spirit of my research is primarily theoretical, I continually seek out potential applications of it to other fields.

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(I) Already posed by Jordanus de Nemore (Jordan of the Forest) in the XIIIth Century.
(II) Also appears in a fundamental problem by G. Galilei (1638).
(III) History can be found in a report by R. Levien (2008).

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- L. Euler (1744): Described the shape of planar elasticae (partially solved by Jacob Bernoulli, 1692-1694).

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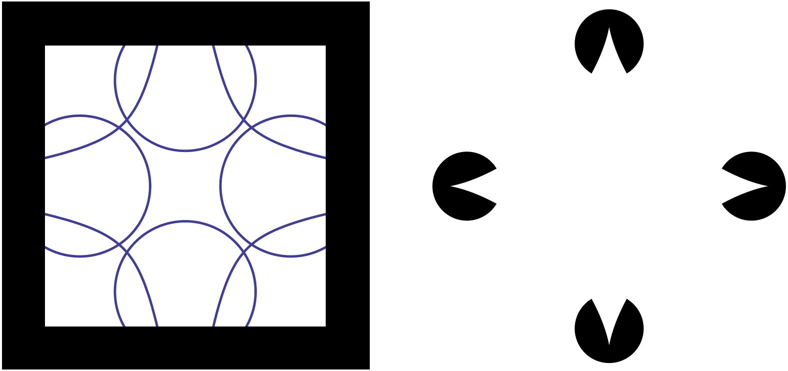
- Applications:
  (I) Image Reconstruction
  (II) Submanifold Theory

Image Reconstruction



(Arroyo, Garay & P., 2016)

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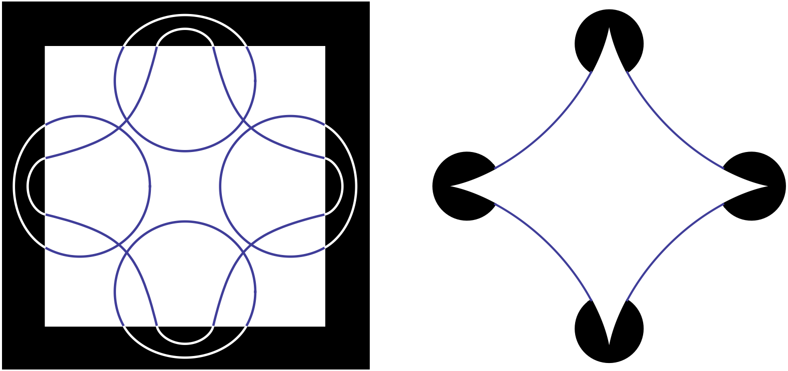

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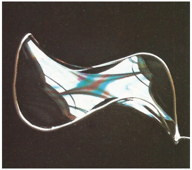

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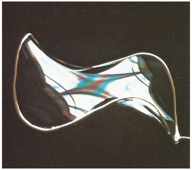

- J. Douglas and T. Radó (1930-1931): Found the general solution to Plateau's problem, independently.

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- H. C. Wente (1894): Found an immersed torus with CMC.

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- F. C. Marques and A. Neves (2012): Proved the Willmore conjecture.

Modeling Biological Membranes

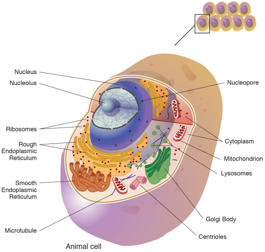
- P. B. Canham (1970): Proposed the minimization of the Willmore energy as a possible explanation for the biconcave shape of red blood cells.

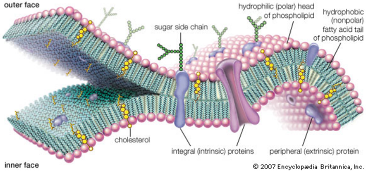
Modeling Biological Membranes

- W. Helfrich (1973): Based on liquid cristallography, suggested the extension

$$\mathcal{H}[\Sigma] := \int_{\Sigma} \left(a\,[H + c_o]^2 + bK \right) d\Sigma\,,$$

to model biological membranes.





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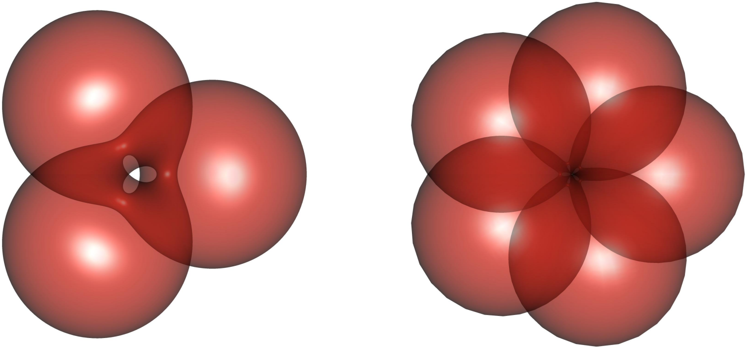
- The Kirchhoff-Plateau Problem. Minimize the area functional where the boundary components are subjected to bending and twisting.
- The Euler-Helfrich Problem. Minimize the Helfrich energy where the boundary components are elastic, i.e.,

$$\mathcal{EH}[\Sigma] := \int_\Sigma \left(a[H + c_o]^2 + bK\right) d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \beta\right) ds\,.$$

Results and Open Problems



(Arroyo, Garay & P., 2018)

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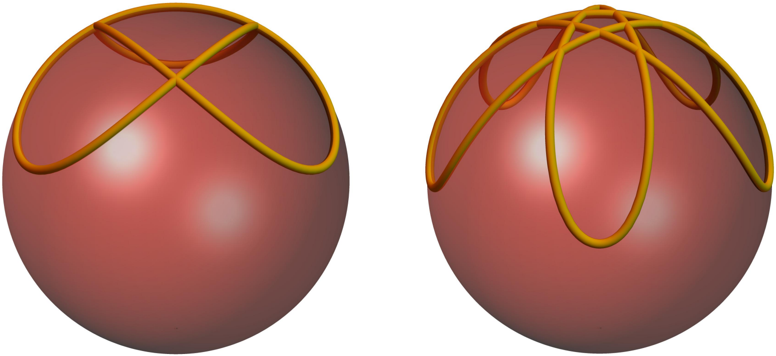

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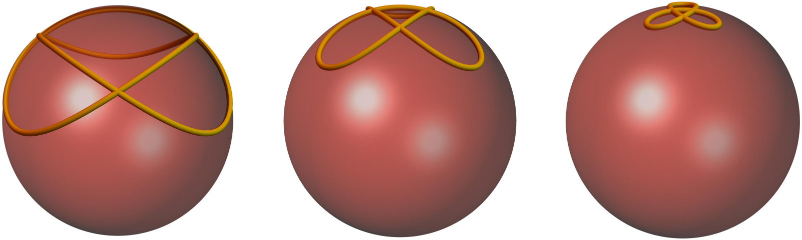

(Oniciuc, Montaldo & P., 2022)

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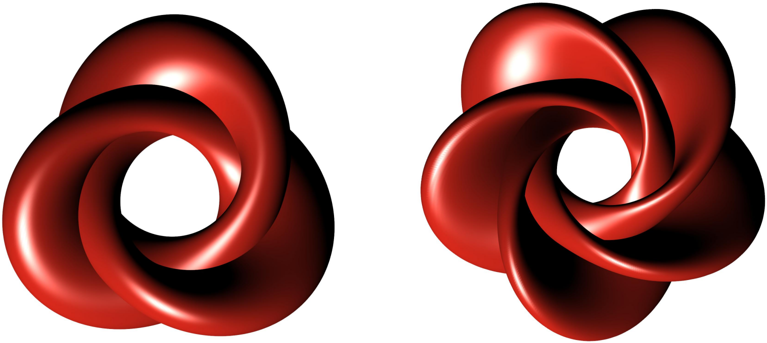

(P., 2020)
(Gruber, P. & Toda, Submitted)

Results and Open Problems

(Palmer & P., 2022)

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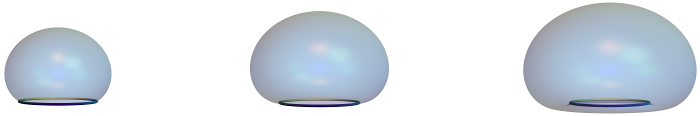

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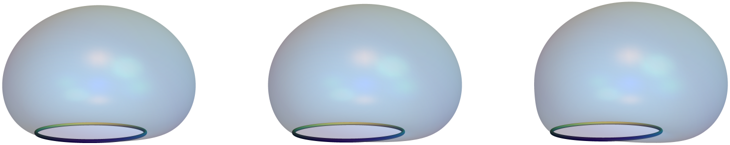

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THE END

Thank You!

https://www.math.ttu.edu/∼apampano/