# LECTURE NOTES

## Math 6331, Riemannian Geometry

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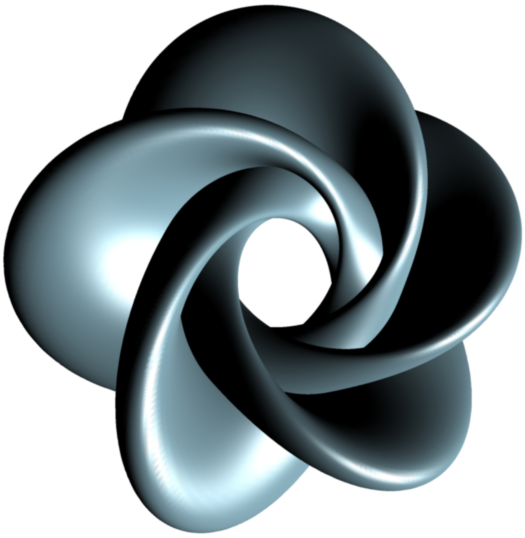

# 1 Preliminaries: Smooth Manifolds

(For more details, see Chapters 1-2 of [8] and/or Chapters 1-3 of [10].)

**Definition 1.1** *Let $M$ be a set of points. A collection of subsets $\tau \subseteq \mathcal{P}(M)$ is a* <u>*topology*</u> *on the set $M$ if:*

*(i) Both the empty set $\emptyset$ and $M$ belong to $\tau$,*

*(ii) Any arbitrary (finite or infinite) union of elements of $\tau$ belongs to $\tau$, and*

*(iii) The intersection of any finite number of elements of $\tau$ belongs to $\tau$.*

*Stereographic projection of the Hopf torus in $\mathbb{S}^3$ based on a critical point $\gamma$ in $\mathbb{S}^2$ of the Blaschke's functional. More details can be found in Exercise 4 of Section 6.4 and in [16]. (This figure was obtained from: Website.)

*The elements of $\tau$ are called open sets. A set is said closed if its complement is an open set. The pair $(M, \tau)$ is a topological space.*

**Remark 1.2** *For convenience, we will simply denote topological spaces by their set $M$ (the topology $\tau$ will always be clearly understood). Roughly speaking, the topology describes all the open sets of $M$.*

**Definition 1.3** *A topological space $M$ is said to be:*

*(i) A Hausdorff space (or T2 space) if any two distinct points can be separated by disjoint open subsets.*

*(ii) A second-countable space if its topology has a countable basis, that is, a countable collection of open subsets such that any arbitrary open subset (in this collection or not) can be written as a union of these basic subsets.*

*(iii) A topological manifold if it is Hausdorff, second-countable, and locally Euclidean, that is, there exists a natural number $n$ (called the dimension of $M$) such that every point $p \in M$ has a neighborhood homeomorphic[1] to an open subset of $\mathbb{R}^n$. To explicitly denote the dimension of a topological manifold we will often write $M^n$.*

**Remark 1.4** *Although the definition of a topological manifold $M$ does not require connectedness, throughout this course we will assume that all the manifolds under consideration are connected (if needed, restricting ourselves to the connected components). Due to the locally Euclidean property of topological manifolds $M$, being connected means that every pair of points in $M$ can be joined by a path in $M$.*

**Definition 1.5** *For each point $p \in M$ in a topological manifold, we have an open neighborhood $U$ of $p$ and a homeomorphism $x : U \subseteq M \longrightarrow x(U) \subseteq \mathbb{R}^n$. The pair $(U, x)$ is called a local chart (or, coordinate chart) on $M$. The component functions $(x_1, ..., x_n)$ of $x$ are called local coordinates. The inverse map $x^{-1} : x(U) \subseteq \mathbb{R}^n \longrightarrow U \subseteq M$ is a local parameterization of the subset $U$ of $M$.*

**Definition 1.6** *Let $M$ be a topological manifold. A smooth atlas $\mathfrak{A}$ for $M$ is a collection of local charts $\mathfrak{A} = \{(U_\alpha, x_\alpha) \,|\, \alpha \in \mathcal{I}\}$ such that $\mathfrak{A}$ covers the whole $M$, that is, $M = \cup_{\alpha \in \mathcal{I}} U_\alpha$, and such that for all $\alpha, \beta \in \mathcal{I}$ the transition maps*

$$x_\beta \circ x_\alpha^{-1}|_{x_\alpha(U_\alpha \cap U_\beta)} \colon x_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n \longrightarrow x_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n \,,$$

*are smooth.*

1. An homeomorphism is a continuous invertible map with continuous inverse. Two topological spaces are homeomorphic if there exists an homeomorphism between them.

**Remark 1.7** *Of course, we can consider atlases with less regularity by allowing transition maps that are only of class $\mathcal{C}^r$. However, throughout this course we will only consider the smooth case.*

**Definition 1.8** *A local chart $(U, x)$ on $M$ is compatible with a smooth atlas $\mathfrak{A}$ on $M$ if $\mathfrak{A} \cup \{(U, x)\}$ is also a smooth atlas. A smooth atlas $\mathfrak{A}$ is maximal if it contains all the local charts compatible with it.*

**Definition 1.9** *A smooth manifold (or, differentiable manifold) is a topological manifold $M$ together with a maximal smooth atlas.*

**Remark 1.10** *The maximal atlas defines a smooth structure on the topological manifold and allows us to employ the techniques coming from calculus.*

**Example 1.11** *In $\mathbb{R}^{n+1}$ define the hyperquadric*

$$\mathbb{S}^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_1^2 + ... + x_{n+1}^2 = 1\} \,.$$

*Use the stereographic projections from the north pole $(0, ..., 0, 1)$ and the south pole $(0, ..., 0, -1)$ to obtain a smooth atlas on $\mathbb{S}^n$. The smooth manifold consisting of $\mathbb{S}^n$ and the atlas constructed this way is called the standard $n$-dimensional sphere.*

## 1.1 Diffeomorphisms

**Remark 1.12** *There are "different" smooth structures that can be defined over $\mathbb{S}^n$. These are called exotic spheres. For instance, the 7-dimensional sphere $\mathbb{S}^7$ has exactly 28 "different" smooth structures (Milnor et al.).*

**Definition 1.13** *Let $M^n$ and $\widetilde{M}^m$ be two smooth manifolds with atlases $\mathfrak{A}$ and $\widetilde{\mathfrak{A}}$, respectively. A map $\phi : M^n \longrightarrow \widetilde{M}^m$ is smooth (or, differentiable) if for all local charts $(U, x) \in \mathfrak{A}$ and $(\widetilde{U}, \widetilde{x}) \in \widetilde{\mathfrak{A}}$ the maps*

$$\widetilde{x} \circ \phi \circ x^{-1}|_{x(U \cap \phi^{-1}(\widetilde{U}))} : x(U \cap \phi^{-1}(\widetilde{U})) \subseteq \mathbb{R}^n \longrightarrow \widetilde{x}(\widetilde{U} \cap \phi(U)) \subseteq \mathbb{R}^m \,,$$

*are smooth.*

**Definition 1.14** *Two smooth manifolds are said to be diffeomorphic if there exists a smooth map, with smooth inverse, between them. Such a map is called a diffeomorphism.*

**Definition 1.15** *A smooth map $f : M \longrightarrow \mathbb{R}$ is a smooth function on $M$. The set of all smooth functions on $M$ is denoted by $\mathcal{C}^\infty(M)$.*

## 1.2 The Tangent Bundle

**Remark 1.16** *Note that the standard sphere* $\mathbb{S}^n$ *defined in Example 1.11 may be defined without involving the ambient space* $\mathbb{R}^{n+1}$*. Nevertheless, we can define its tangent space.*

**Definition 1.17** *Let* $M$ *be a smooth manifold and* $p \in M$*. A* tangent vector $X_p$ *at* $p \in M$ *is a* derivation*, that is, a map* $X_p : \mathcal{C}^\infty(M) \longrightarrow \mathbb{R}$ *satisfying:*

*(i) Linearity:* $X_p\,(af + bg) = aX_p(f) + bX_p(g)$,

*(ii) Leibniz Property:* $X_p(fg) = X_p(f)\,g(p) + f(p)X_p(g)$,

*for all* $a, b \in \mathbb{R}$ *and* $f, g \in \mathcal{C}^\infty(M)$*. The set of all tangent vectors at* $p$ *is called the* tangent space *at* $p$ *and it is denoted by* $T_pM$*.*

**Definition 1.18** *Let* $\phi : M \longrightarrow \widetilde{M}$ *be a smooth map. The* differential $d\phi_p$ *of* $\phi$ *at* $p \in M$ *is the linear map* $d\phi_p : T_pM \longrightarrow T_{\phi(p)}\widetilde{M}$ *such that for all* $X_p \in T_pM$ *and* $f \in \mathcal{C}^\infty(\widetilde{M})$,

$$d\phi_p(X_p)[f] = X_p(f \circ \phi)\,.$$

**Definition 1.19** *Let* $M^n$ *be a smooth manifold. Define the set*

$$TM = \{(p, X_p)\,|\, p \in M, X_p \in T_pM\}\,,$$

*and consider the projection map* $\pi : TM \longrightarrow M$ *given by* $\pi(p, X_p) = p$*. In a standard way,* $TM$ *is a smooth manifold of dimension* $2n$*. Moreover, the triple* $(TM, M, \pi)$ *is a smooth vector bundle*[2] *called the* tangent bundle *of* $M$*.*

**Definition 1.20** *A* smooth vector field *is a section of the tangent bundle, that is, a smooth map* $X : M \longrightarrow TM$ *such that* $\pi(X(p)) = p$ *for all* $p \in M$*. The set of all smooth vector fields is denoted by* $\mathfrak{X}(M)$*.*

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2. A (real) vector bundle is a triple $(E, M, \pi)$ where:

(i) $E$ and $M$ are topological spaces, known as total space and base space, respectively,

(ii) $\pi : E \longrightarrow M$ is a continuous surjective map, called the bundle projection, and

(iii) For every $p \in M$, $\pi^{-1}(p)$ is a finite-dimensional (real) vector space,

such that the following compatibility condition is satisfied: for every $p \in M$, there exists an open neighborhood $U \subseteq M$ of $p$, a natural number $n$, and an homeomorphism $\varphi : U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$, such that for all $p \in U$:

1. $\pi \circ \varphi(p, v) = p$ for all $v \in \mathbb{R}^n$,
2. The map $v \in \mathbb{R}^n \longmapsto \varphi(p, v) \in \pi^{-1}(U)$ is a linear isomorphism.

**Remark 1.21** *In other words, a smooth vector field $X$ on $M$ is a smooth assignment of a tangent vector $X_p$ to each point $p \in M$.*

**Remark 1.22** *Let $(U, x)$ be a local chart on a smooth manifold $M^n$. This defines an isomorphism $T_pM \cong \mathbb{R}^n$ for every $p \in U$. The tangent space $T_pM$ at $p \in U$ is then spanned by the coordinate (or, canonical) basis $\{\partial_{x_1}|_p, ..., \partial_{x_n}|_p\}$. We call $\partial_{x_i}|_p$, $i = 1, ..., n$, the coordinate vectors at $p \in U$. The set $\{\partial_{x_1}, ..., \partial_{x_n}\}$ is a local frame of the tangent bundle, defined by the property that for each $p \in U$, their restriction gives the coordinate basis.*

# 2 Riemannian Manifolds

(For more details, see Chapters 1-2 and 4 of [5] and/or Chapters 2-5 and 7 of [9].)

## 2.1 Riemannian Metrics

**Definition 2.1** *Let $M$ be a smooth manifold. A* <u>*symmetric tensor field*</u> *$\Phi$ on $M$ of type $(0, 2)$ is a map $\Phi : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{C}^\infty(M)$ satisfying:*

*(i)* $\Phi(X, Y) = \Phi(Y, X)$,

*(ii)* $\Phi(X, fY + gZ) = f\Phi(X, Y) + g\Phi(X, Z)$,

*for all $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in \mathcal{C}^\infty(M)$.*

**Definition 2.2** *Let $M$ be a smooth manifold. A* <u>*Riemannian metric*</u> *$g$ is a symmetric positive definite*[3] *tensor field of type $(0, 2)$.*

**Remark 2.3** *A Riemannian metric $g$ on $M$ determines an inner product $g_p$ at each tangent space $T_pM$, which varies smoothly from point to point. When necessary, we will denote the Riemannian metric, simply, by $g \equiv \langle \cdot, \cdot \rangle$.*

**Definition 2.4** *A* <u>*Riemannian manifold*</u> *is a smooth manifold $M$ endowed with a Riemannian metric.*

**Remark 2.5** *If we relax the requirement that the symmetric tensor field $g$ of type $(0, 2)$ is definite positive and we simply ask that it is everywhere non-degenerate, we obtain that $M$ endowed with $g$ is a* <u>*pseudo-Riemannian*</u> *(or,* <u>*semi-Riemannian*</u>*)* <u>*manifold*</u>*, [3, 14]. These manifolds are the framework for relativity theory, [14].*

**Theorem 2.6** *Every smooth manifold admits a Riemannian metric.*

**Example 2.7** *The followings are examples of Riemannian manifolds:*

*(i) Consider the space $\mathbb{R}^n$ with the standard coordinates $(x_1, ..., x_n)$. Define the Euclidean metric $\bar{g}$ by*

$$\bar{g} = \sum_{i=1}^{n} dx_i^2 \, .$$

*Then, $\mathbb{R}^n$ endowed with this Riemannian metric is the* <u>*Euclidean space*</u> *of dimension $n$.*

3. A tensor field $\Phi$ of type $(0, 2)$ on $M$ is <u>positive definite</u> if $\Phi(X, X) \geq 0$ for all $X \in \mathfrak{X}(M)$ and with equality holding if and only if $X = 0$ identically.

*(ii) Let $\mathbb{S}^n$ be the standard $n$-dimensional sphere defined in Example 1.11 and consider the restriction*[4] *of the Euclidean metric $\bar{g}$ to the vector fields tangent to $\mathbb{S}^n$. The sphere $\mathbb{S}^n$ endowed with this metric is the* <u>*round sphere*</u> *of dimension $n$.*

*(iii) Consider the (interior of the) ball of radius one in $\mathbb{R}^n$, that is, the set of points $(x_1, ..., x_n) \in \mathbb{R}^n$ satisfying $x_1^2 + ... + x_n^2 < 1$, with the metric*

$$g = \frac{4}{\left(1 - \sum_{i=1}^n x_i^2\right)^2} \sum_{i=1}^n dx_i^2 \,.$$

*This is the Poincaré disc model of the $n$-dimensional* <u>*hyperbolic space*</u> *$\mathbb{H}^n$.*

## 2.2 Isometries

**Remark 2.8** *Consider the standard sphere of dimension two $\mathbb{S}^2$ defined in Example 1.11 with the spherical coordinates $(\theta, \varphi)$, where $\theta$ represents the longitude and $\varphi$ is colatitude. Together with the metric*

$$g = d\varphi^2 + \sin^2 \varphi \, d\theta^2$$

*we have the 2-dimensional round sphere. (It is "the same" as in previous example.)*

**Definition 2.9** *Let $(M, g)$ and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds. A diffeomorphism $\phi : M \longrightarrow \widetilde{M}$ is said to be an* <u>*isometry*</u> *if $\phi^* \widetilde{g} = g$. In other words, if*

$$g_{|_p}(X_p, Y_p) = \widetilde{g}_{|_{\phi(p)}}(d\phi_p(X_p), d\phi_p(Y_p))$$

*for all $X, Y \in \mathfrak{X}(M)$ and $p \in M$. (Recall that $d\phi_p$ is the differential defined in Definition 1.18.)*

**Definition 2.10** *Two Riemannian manifolds are* <u>*isometric*</u> *if there exists an isometry between them.*

**Remark 2.11** *Among the class of Riemannian manifolds, being isometric defines an equivalence relation.*

## 2.3 The Levi-Civita Connection

**Definition 2.12** *Let $M$ be a smooth manifold. An* <u>*affine connection*</u> *on $M$ is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ defined by $(X, Y) \longmapsto \nabla_X Y$ such that:*

4. The restriction of a Riemannian metric on a smooth manifold to a submanifold is a standard procedure which we will see in detail in Chapter 2.

*(i) The map* $\nabla$ *is lineal in* $X$ *over* $\mathcal{C}^\infty(M)$*, that is,*

$$\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z\,,$$

*for all* $f, g \in \mathcal{C}^\infty(M)$*.*

*(ii) The map* $\nabla$ *is lineal in* $Y$ *over* $\mathbb{R}$*, that is,*

$$\nabla_X (aY + bZ) = a\nabla_X Y + b\nabla_X Z\,,$$

*for all* $a, b \in \mathbb{R}$*.*

*(iii) The map* $\nabla$ *satisfies the product rule*

$$\nabla_X (fY) = f\nabla_X Y + (Xf)\,Y\,,$$

*for all* $f \in \mathcal{C}^\infty(M)$*.*

*The vector field* $\nabla_X Y$ *is referred to as the* *covariant derivative* *of* $Y$ *in the direction of* $X$*.*

**Remark 2.13** *The concept of affine connection is local.*

**Definition 2.14** *An affine connection* $\nabla$ *defined over a smooth manifold is* *symmetric* *(or,* *torsion-free**) if*

$$\nabla_X Y - \nabla_Y X = [X, Y]\,,$$

*where* $[\cdot,\cdot]$ *represents the Lie bracket (or, commutator) of vector fields, which is defined by*

$$[X, Y](f) = X\,(Yf) - Y\,(Xf)\,,$$

*for all* $f \in \mathcal{C}^\infty(M)$*.*

**Remark 2.15** *If* $(U, x)$ *is a local chart for the smooth manifold* $M^n$*, the fact that the affine connection* $\nabla$ *is symmetric implies that*

$$\nabla_{\partial_{x_i}}\partial_{x_j} - \nabla_{\partial_{x_j}}\partial_{x_i} = [\partial_{x_i}, \partial_{x_j}] = 0\,,$$

*for all* $i, j = 1, ..., n$*. This is the reason why such a connection is called symmetric.*

**Remark 2.16** *An affine connection is defined over a smooth manifold. The Riemannian metric does not play any role yet. However, we will see that for Riemannian manifolds there is a preferred choice of affine connection.*

**Definition 2.17** *Let* $M$ *be a Riemannian manifold. An affine connection* $\nabla$ *is* *compatible* *with the Riemannian metric* $g \equiv \langle\cdot,\cdot\rangle$ *if it satisfies*

$$X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle\,,$$

*for all* $X, Y, Z \in \mathfrak{X}(M)$*.*

**Theorem 2.18 (Fundamental Theorem of Riemannian Geometry)** *For every Riemannian manifold $M$, there exists a unique symmetric affine connection $\nabla$ defined over $M$ which is compatible with the metric.*

**Definition 2.19** *The unique symmetric affine connection compatible with the Riemannian metric is called the* <u>*Levi-Civita connection*</u>.

**Remark 2.20** *The Levi-Civita connection is given by the Koszul formula*

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle + \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle - \langle [X,Z], Y\rangle\right).$$

**Remark 2.21** *Let $(U,x)$ be a local chart for the Riemannian manifold $M^n$ and consider the local frame $\{\partial_{x_i}\}_{i=1}^n$ of the tangent bundle. Then,*

$$\nabla_{\partial_{x_i}} \partial_{x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \, \partial_{x_k} \,,$$

*for all $i, j = 1, ..., n$. Hence, the Levi-Civita connection $\nabla$ defines $n^3$ functions $\Gamma_{ij}^k : U \subseteq M^n \longrightarrow \mathbb{R}$. From Koszul formula we deduce the explicit expressions*

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{m=1}^{n}\left(\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{mi}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m}\right) g^{mk} \,,$$

*where $g_{ij} = \langle \partial_{x_i}, \partial_{x_j}\rangle$ and $g^{mk} = \left(g^{-1}\right)_{mk}$.*

**Definition 2.22** *The functions $\Gamma_{ij}^k$ defined above are the* <u>*Christoffel symbols*</u> *of the connection $\nabla$ with respect to the local chart $(U,x)$.*

**Example 2.23** *Compute the Christoffel symbols of the Riemannian manifolds of Example 2.7 for dimension two.*

## 2.4 Curvature

**Definition 2.24** *Let $M$ be a Riemannian manifold with Levi-Civita connection $\nabla$. The (Riemann)* <u>*curvature tensor*</u> *is the map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ defined by*

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \,,$$

*where $[\cdot,\cdot]$ is the Lie bracket.*

**Remark 2.25** *The Riemann curvature tensor may be found in the literature with the opposite sign.*

**Remark 2.26** *Let $(U, x)$ be a local chart for $M^n$ and consider the local frame $\{\partial_{x_i}\}_{i=1}^n$ of the tangent bundle. Then,*

$$R\left(\partial_{x_i}, \partial_{x_j}\right)\partial_{x_k} = \left(\nabla_{\partial_{x_j}}\nabla_{\partial_{x_i}} - \nabla_{\partial_{x_i}}\nabla_{\partial_{x_j}}\right)\partial_{x_k}\,,$$

*for all $i, j, k = 1, ..., n$. Hence, the Riemann curvature tensor measures the non-commutativity of the covariant derivative.*

**Remark 2.27** *The Riemann curvature tensor is a tensor field of type $(1, 3)$. Employing the Riemann metric, we may identify tensor fields of type $(1, 3)$ with tensor fields of type $(0, 4)$.*

**Definition 2.28** *The Riemann curvature of a Riemannian manifold $(M, g \equiv \langle\cdot,\cdot\rangle)$ is the tensor field of type $(0, 4)$ defined by*

$$\mathrm{Rm}(X, Y, Z, T) = \langle R(X, Y)Z, T\rangle\,,$$

*for all $X, Y, Z, T \in \mathfrak{X}(M)$ and where $R$ is the Riemann curvature tensor.*

**Proposition 2.29 (Bianchi's First Identity)** *Let $\mathrm{Rm}$ be the Riemann curvature of a Riemannian manifold $M$. Then,*

$$\mathrm{Rm}(X, Y, Z, T) + \mathrm{Rm}(Y, Z, X, T) + \mathrm{Rm}(Z, X, Y, T) = 0\,,$$

*for all $X, Y, Z, T \in \mathfrak{X}(M)$.*

**Proposition 2.30** *Let $M$ be a Riemannian manifold and $X, Y, Z, T \in \mathfrak{X}(M)$. The Riemann curvature $\mathrm{Rm}$ satisfies the following properties:*

*(i)* $\mathrm{Rm}(X, Y, Z, T) = -\mathrm{Rm}(Y, X, Z, T)$.

*(ii)* $\mathrm{Rm}(X, Y, Z, T) = -\mathrm{Rm}(X, Y, T, Z)$.

*(iii)* $\mathrm{Rm}(X, Y, Z, T) = \mathrm{Rm}(Z, T, X, Y)$.

**Remark 2.31** *Even though the Riemann curvature tensor $R$ may be defined with opposite sign, the choice of definition and the above symmetries of the Riemann curvature $\mathrm{Rm}$ will make them coincide.*

**Definition 2.32** *The Ricci tensor $\mathrm{Ric}$ is the tensor field of type $(0, 2)$ defined as the trace of the Riemann curvature $\mathrm{Rm}$ in the second and last indexes.*

**Remark 2.33** *In other words, let $X, Y \in \mathfrak{X}(M)$ and $\{e_i\}_{i=1}^n$ be any local orthonormal (that is, $\langle e_i, e_j\rangle = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker delta[5]) frame for the tangent bundle. Then,*

$$\operatorname{Ric}(X,Y) = \sum_{i=1}^n \operatorname{Rm}(X, e_i, Y, e_i) = \sum_{i=1}^n \langle R(X, e_i)Y, e_i\rangle\,.$$

*It follows from the symmetries of the Riemann curvature* Rm *that the Ricci tensor is symmetric, that is,* $\operatorname{Ric}(X,Y) = \operatorname{Ric}(Y,X)$.

**Definition 2.34** *Let $M$ be a Riemannian manifold and consider a unitary tangent vector $X_p \in T_pM$ of the tangent space at $p \in M$. The* <u>*Ricci curvature*</u> *is defined by*

$$\operatorname{Ric}_p(X_p) = \operatorname{Ric}(X_p, X_p)\,.$$

**Definition 2.35** *Let $M$ be a Riemannian manifold and $\pi \subseteq T_pM$ be a two-dimensional linear subspace spanned by $X_p, Y_p \in \pi$. The* <u>*sectional curvature*</u> *of $\pi$ at $p \in M$ is*

$$K(\pi) = \frac{\operatorname{Rm}(X_p, Y_p, X_p, Y_p)}{\langle X_p, X_p\rangle\langle Y_p, Y_p\rangle - \langle X_p, Y_p\rangle^2}\,.$$

**Proposition 2.36** *For every $p \in M$ and every two-dimensional linear subspace $\pi \subseteq T_pM$, the sectional curvature $K(\pi)$ is independent of the choice of bases for $\pi$.*

**Remark 2.37** *Knowing the sectional curvature of every two-dimensional linear subspace is enough to recover the Riemann curvature* Rm*. First, we need to compute* $\operatorname{Rm}(X+Z, Y, X+Z, Y)$ *and then* $\operatorname{Rm}(X+Z, Y+T, X+Z, Y+T)$*.*

**Example 2.38** *Compute the sectional curvature of the Riemannian manifolds of Example 2.7 for dimension two.*

**Remark 2.39** *If the Riemannian manifold is a regular surface, the sectional curvature is just the classical Gaussian curvature. Details will be explained in the next chapter.*

**Definition 2.40** *The* <u>*scalar curvature*</u> *$\lambda$ of a Riemannian manifold is the function defined as the trace of the Ricci tensor.*

**Remark 2.41** *Let $(U, x)$ be a local chart for $M$ and $\{e_i\}_{i=1}^n$ any local orthonormal frame for the tangent bundle. Then, the scalar curvature $\lambda \in \mathcal{C}^\infty(M)$ is given by*

$$\lambda(p) = \sum_{i=1}^n \operatorname{Ric}(e_i, e_i) = \sum_{i=1}^n \sum_{j=1}^n \operatorname{Rm}(e_i, e_j, e_i, e_j)\,.$$

5. The <u>Kronecker delta</u> $\delta_{ij}$ is defined to be 1 if $i = j$, and 0 otherwise.

## 2.5 Exercises

1. * Let $\mathcal{H} = \{(u,v) \in \mathbb{R}^2 \,|\, v > 0\}$ be the upper half-plane endowed with the metric
$$g = \frac{1}{v^2}\left(du^2 + dv^2\right).$$
If $\mathbb{H}^2$ denotes the hyperbolic plane (defined as in Example 2.7), prove that the map $\phi : \mathbb{H}^2 \longrightarrow \mathcal{H}$ given by
$$\phi : (x,y) \longmapsto \frac{1}{x^2 + (y-1)^2}\left(2x, 1 - x^2 - y^2\right),$$
is an isometry. The Riemannian manifold $(\mathcal{H}, g)$ is the upper half-plane model for the hyperbolic plane.

2. * Let $(M^n, g)$ be a Riemannian manifold and $f \in \mathcal{C}^\infty(M)$. The gradient of $f$ is the vector field $\operatorname{grad} f$ defined by the property
$$df_p(X_p) = \langle \operatorname{grad}_p f, X_p \rangle,$$
for all $p \in M^n$ and $X_p \in T_pM^n$.

   i) Show that, in the local coordinates $(U, x)$ the expression of $\operatorname{grad} f$ is
$$\operatorname{grad} f = \sum_{i=1}^n \left( \sum_{j=1}^n g^{ij} \frac{\partial f}{\partial x_j} \right) \partial_{x_i}.$$

   ii) Prove that if $\{e_i\}_{i=1}^n$ is any local orthonormal frame then the components of $\operatorname{grad} f$ are the same of the differential $df$.

   iii) Show that if $M^n = \mathbb{R}^n$ is the Euclidean space (defined in Example 2.7), then
$$\operatorname{grad} f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \partial_{x_i}.$$

3. * Let $(M, g)$ be a Riemannian manifold, $X \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty(M)$. We define the divergence of $X$ as the smooth function $\operatorname{div} X : M \longrightarrow \mathbb{R}$ so that $\operatorname{div}_p X$ is given by the trace of the linear map $Y_p \longmapsto \nabla_Y X|_p$ for every $p \in M$. The Laplacian of $M$ is the operator $\Delta : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ defined by
$$\Delta f = \operatorname{div}(\operatorname{grad} f),$$
for every $f \in \mathcal{C}^\infty(M)$, where $\operatorname{grad} f$ is the gradient of $f$ (see the previous exercise).

   i) Show that
$$\Delta(f \cdot g) = f\Delta g + g\Delta f + 2\langle \operatorname{grad} f, \operatorname{grad} g \rangle,$$
   holds for all $f, g \in \mathcal{C}^\infty(M)$.

ii) Let $(U, x)$ be a local chart for $M$ and consider the local frame $\{\partial_{x_i}\}_{i=1}^n$ of the tangent bundle. Show that the divergence and the Laplacian are given in these local coordinates by

$$\operatorname{div} X = \operatorname{div}\left(\sum_{i=1}^n X_i \partial_{x_i}\right) = \frac{1}{\sqrt{\det g}} \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(X_i \sqrt{\det g}\right),$$

and

$$\Delta f = \frac{1}{\sqrt{\det g}} \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n g^{ij} \sqrt{\det g}\, \frac{\partial f}{\partial x_j}\right).$$

iii) Prove that if $M = \mathbb{R}^n$ is the Euclidean space (defined in Example 2.7), then

$$\operatorname{div} X = \operatorname{div}\left(\sum_{i=1}^n X_i \partial_{x_i}\right) = \sum_{i=1}^n \frac{\partial X_i}{\partial x_i},$$

and

$$\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}.$$

4. * Let $M$ be a Riemannian manifold and $X \in \mathfrak{X}(M)$. Consider a smooth map $\phi : (-\epsilon, \epsilon) \times U \longrightarrow M$, where $U$ is a neighborhood of a point $p \in M$, such that for any $q \in U$, $t \longmapsto \phi(t, q)$ is a trajectory of $X$ passing through $q$ at $t = 0$. The vector field $X \in \mathfrak{X}(M)$ is called a Killing vector field (or, an infinitesimal isometry) if for each $t_o \in (-\epsilon, \epsilon)$, the map $\phi_{t_o} : U \subseteq M \longrightarrow M$ is an isometry. Show that a vector field $X \in \mathfrak{X}(M)$ is a Killing vector field if and only if

$$\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0,$$

for all $Y, Z \in \mathfrak{X}(M)$.

5. Prove Bianchi's First Indentity (Proposition 2.29).

6. Prove the symmetries of the Riemann curvature Rm described in Proposition 2.30.

7. * Let $\mathbb{L}^3$ denote the Lorentz-Minkowski 3-space, that is, $\mathbb{R}^3$ endowed with the Lorentzian metric (semi-Riemannian metric of index one, [3])

$$g = dx^2 + dy^2 - dz^2.$$

Define the quadric $\mathcal{H} = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 - z^2 = -1,\ z > 0\}$ and endow it with the metric obtained by restricting $g$ to $\mathcal{H}$, namely, $g|_{\mathcal{H}}$.

i) Prove that $g|_{\mathcal{H}}$ is a Riemannian metric and so $(\mathcal{H}, g|_{\mathcal{H}})$ is a Riemannian manifold.

ii) Compute the sectional curvature of $(\mathcal{H}, g|_{\mathcal{H}})$.

iii) Prove that $(\mathcal{H}, g|_{\mathcal{H}})$ is isometric to the hyperbolic plane $\mathbb{H}^2$ (defined as in Example 2.7). (Hint: Use the stereographic projection from $(0, 0, -1)$.)

The Riemannian manifold $(\mathcal{H}, g|_{\mathcal{H}})$ is the hyperboloid model for the hyperbolic plane.

8. Let $M$ be a Riemannian manifold of constant sectional curvature $K$. Prove that the Riemann curvature tensor $R$ is given by

$$R(X,Y)Z = K\left(\langle X,Z\rangle Y - \langle Y,Z\rangle X\right),$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

9. Let $M^n$ be a Riemannian manifold of constant sectional curvature $K$. Prove that the scalar curvature $\lambda$ is given by

$$\lambda = n(n-1)K\,.$$

10. * A Riemannian metric is said to be an <u>Einstein metric</u> if at every point the Ricci tensor is a scalar multiple of the metric, that is, if

$$\mathrm{Ric} = f\,g\,,$$

holds for some function $f \in \mathcal{C}^\infty(M)$. A Riemannian manifold whose metric is Einstein is called an <u>Einstein manifold</u>, [2]. Prove that Riemannian manifolds with constant sectional curvature are Einstein manifolds.

# 3 Riemannian Submanifolds

(For more details, see Chapter 6 of [5] and/or Chapter 8 of [9].)

## 3.1 Immersions and Embeddings

**Definition 3.1** *A smooth map $\phi : M \longrightarrow \widetilde{M}$ between smooth manifolds is said to be an* <u>immersion</u> *if for each $p \in M$, the differential*[6] $d\phi_p : T_pM \longrightarrow T_{\phi(p)}\widetilde{M}$ *is injective.*

**Definition 3.2** *An* <u>embedding</u> *is an immersion which is a homeomorphism onto its image.*

**Remark 3.3** *Since every immersion is locally an embedding and our computations are going to be local, throughout this course we may think our maps in either way.*

**Theorem 3.4 (Whitney)** *Every smooth manifold $M^n$ can be embedded into $\mathbb{R}^{2n+1}$.*

**Remark 3.5** *Up to now, we have not used the Riemannian metric.*

**Definition 3.6** *Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold, $M$ a smooth manifold, and $\phi : M \longrightarrow \widetilde{M}$ an immersion. We define the* <u>induced metric</u> *on $M$ as $g = \phi^*\widetilde{g}$. That is,*

$$g_{|_p}(X_p, Y_p) = \widetilde{g}_{|_{\phi(p)}}(d\phi_p(X_p), d\phi_p(Y_p)) \,,$$

*for all $X, Y \in \mathfrak{X}(M)$ and $p \in M$. If $M$ is endowed with the induced metric $g$, then $\phi : M \longrightarrow \widetilde{M}$ is said to be an* <u>isometric immersion</u>.

**Theorem 3.7 (Nash)** *Every Riemannian manifold can be isometrically embedded into $\mathbb{R}^n$, for $n$ sufficiently large.*

**Remark 3.8** *The condition that $n$ is sufficiently large is essential. For instance:*

*(i) Hilbert's Theorem ([4], Section 5-11). Complete regular surfaces of constant negative Gaussian curvature cannot be isometrically embedded in $\mathbb{R}^3$.*

*(ii) Flat tori cannot be isometrically embedded in $\mathbb{R}^3$ since every compact regular surface must have elliptic points, [4].*

*(iii) The projective plane cannot be isometrically embedded in $\mathbb{R}^3$ (even though it has constant positive Gaussian curvature) because it is not orientable ([4], Page 436).*

6. See the definition of the differential (Definition 1.18).

**Definition 3.9** *Let $\phi : M^n \longrightarrow \widetilde{M}^m$ be an isometric immersion. We say that $M^n$ is a Riemannian submanifold of $\widetilde{M}^m$ and $\widetilde{M}^m$ is referred to as the ambient space. The natural number $m-n$ is the codimension.*

**Remark 3.10** *If $M$ is a Riemannian submanifold of $\widetilde{M}$, we will identify $M$ and $\phi(M) \subseteq \widetilde{M}$. We will also denote the metrics $g$ and $\widetilde{g}$, simply, by $\langle \cdot, \cdot \rangle$.*

**Example 3.11** *The round sphere $\mathbb{S}^n$ defined in Example 2.7 is a Riemannian submanifold of $\mathbb{R}^{n+1}$.*

## 3.2 Second Fundamental Form

**Definition 3.12** *Let $M$ be a Riemannian submanifold of $\widetilde{M}$. For every $p \in M$, we call $T_p\widetilde{M}$ the ambient tangent space and*

$$T\widetilde{M}|_M = \{(p, \widetilde{X}_p) \,|\, p \in M, \widetilde{X}_p \in T_p\widetilde{M}\}$$

*the ambient tangent bundle.*

**Remark 3.13** *The ambient tangent bundle $T\widetilde{M}|_M$ and the tangent bundle of the ambient space $T\widetilde{M}$ are not the same. However, every smooth vector field on $\widetilde{M}$ can be restricted to $T\widetilde{M}|_M$ and, conversely, every section of $T\widetilde{M}|_M$ can be locally extended to $T\widetilde{M}$. With some abuse of notation, we will denote both vector fields with the same letter.*

**Remark 3.14** *The metric $\widetilde{g}$ on $\widetilde{M}$ gives us the notion of orthogonality and, hence, we can split $T_p\widetilde{M}$ as the orthogonal direct sum*

$$T_p\widetilde{M} = T_pM \oplus N_pM\,,$$

*where $T_pM$ is the tangent space of $M$ at $p$ and*

$$N_pM = \{\widetilde{X}_p \in T_p\widetilde{M} \,|\, \widetilde{g}(\widetilde{X}_p, Y_p) = 0 \text{ for all } Y_p \in T_pM\}\,.$$

**Definition 3.15** *The vector space $N_pM = (T_pM)^\perp$ is called the normal space of $M$ at $p \in M$. The normal bundle of $M$ in $\widetilde{M}$ is*

$$NM = \{(p, \xi_p) \,|\, p \in M, \xi_p \in N_pM\}\,.$$

*We denote by $\mathfrak{N}(M)$ the set of all smooth vector fields of $\widetilde{M}$ normal to $M$.*

**Remark 3.16** *According to the orthogonal split of the ambient tangent space, we can decompose the covariant derivative as*

$$\widetilde{\nabla}_X Y = \left(\widetilde{\nabla}_X Y\right)^\top + \left(\widetilde{\nabla}_X Y\right)^\perp,$$

*where $\widetilde{\nabla}$ is the Levi-Civita connection of $\widetilde{M}$ and $X, Y \in \mathfrak{X}(M)$ denote as well their local extensions to $\widetilde{M}$ (cf. Remark 3.13).*

**Proposition 3.17** *Let $M$ be a Riemannian submanifold of $\widetilde{M}$ and denote the Levi-Civita connection of $\widetilde{M}$ by $\widetilde{\nabla}$. The affine connection $\nabla$ defined over $M$ by*

$$\nabla_X Y = \left(\widetilde{\nabla}_X Y\right)^{\top},$$

*for every $X, Y \in \mathfrak{X}(M)$ and their arbitrary local extensions, is the Levi-Civita connection of $M$.*

**Definition 3.18** *Let $M$ be a Riemannian submanifold of $(\widetilde{M}, \widetilde{\nabla})$. The* <u>*second fundamental form*</u> *of $M$ is the map $\mathrm{II} : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{N}(M)$ given by*

$$\mathrm{II}(X, Y) = \left(\widetilde{\nabla}_X Y\right)^{\perp},$$

*where $X, Y \in \mathfrak{X}(M)$ denote as well their arbitrary local extensions to $\widetilde{M}$.*

**Proposition 3.19** *The second fundamental form $\mathrm{II}$ is independent of the extensions of $X, Y \in \mathfrak{X}(M)$ and it is symmetric, that is,*

$$\mathrm{II}(X, Y) = \mathrm{II}(Y, X),$$

*for all $X, Y \in \mathfrak{X}(M)$.*

**Theorem 3.20 (Gauss Formula)** *Let $(M, \nabla)$ be a Riemannian submanifold of $(\widetilde{M}, \widetilde{\nabla})$. Then,*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X, Y),$$

*for all $X, Y \in \mathfrak{X}(M)$ and their arbitrary local extensions.*

**Remark 3.21** *Let $X \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{N}(M)$. The covariant derivative $\widetilde{\nabla}_X \xi$ in $\widetilde{M}$ can be decomposed into the tangential and normal components as*

$$\widetilde{\nabla}_X \xi = \left(\widetilde{\nabla}_X \xi\right)^{\top} + \left(\widetilde{\nabla}_X \xi\right)^{\perp}.$$

*We denote by $\mathcal{D}_X \xi$ to the normal component, which defines an affine connection on the normal bundle $NM$.*

**Definition 3.22** *Let $M$ be a Riemannian submanifold of $(\widetilde{M}, \widetilde{\nabla})$. For every $\xi \in \mathfrak{N}(M)$, we define the* <u>*Weingarten endomorphism*</u> *$\mathcal{A}_\xi : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ by*

$$\mathcal{A}_\xi X = -\left(\widetilde{\nabla}_X \xi\right)^{\top},$$

*for every $X \in \mathfrak{X}(M)$.*

**Theorem 3.23 (Weingarten Formula)** *Let $M$ be a Riemannian submanifold of $\widetilde{M}$. Then,*

$$\widetilde{\nabla}_X \xi = -\mathcal{A}_\xi X + \mathcal{D}_X \xi \,,$$

*for all $X \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{N}(M)$.*

**Theorem 3.24 (Weingarten Equation)** *Let $M$ be a Riemannian submanifold of $\widetilde{M}$ and consider $X, Y \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{N}(M)$. Then,*

$$\langle \widetilde{\nabla}_X \xi, Y \rangle = -\langle \mathrm{II}(X, Y), \xi \rangle \,,$$

*holds along $M$, where $\mathrm{II}$ is the second fundamental form.*

**Remark 3.25** *Employing the Weingarten formula, the Weingarten equation may be rewritten as*

$$\langle \mathcal{A}_\xi X, Y \rangle = \langle \mathrm{II}(X, Y), \xi \rangle \,.$$

*We then deduce from the symmetry of the second fundamental form $\mathrm{II}$ that the Weingarten endomorphism $\mathcal{A}_\xi$ is self-adjoint, that is,*

$$\langle \mathcal{A}_\xi X, Y \rangle = \langle X, \mathcal{A}_\xi Y \rangle \,.$$

## 3.3 Fundamental Equations

**Theorem 3.26 (Gauss Equation)** *Let $M$ be a Riemannian submanifold of $\widetilde{M}$. Then,*

$$\widetilde{\mathrm{Rm}}(X, Y, Z, T) = \mathrm{Rm}(X, Y, Z, T) - \langle \mathrm{II}(X, Z), \mathrm{II}(Y, T) \rangle + \langle \mathrm{II}(Y, Z), \mathrm{II}(X, T) \rangle \,,$$

*for all $X, Y, Z, T \in \mathfrak{X}(M)$.*

*Proof.* Let $X, Y, Z \in \mathfrak{X}(M)$ and assume they are arbitrarily extended to vector fields on $\widetilde{M}$ tangent to $M$. Then, from the definition of the Riemann curvature tensors of $\widetilde{M}$ and $M$, $\widetilde{R}$ and $R$ respectively, and Gauss formula, we have

$$\begin{aligned}
\widetilde{R}(X, Y)Z &= \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_X \widetilde{\nabla}_Y Z + \widetilde{\nabla}_{[X,Y]} Z \\
&= \widetilde{\nabla}_Y \left( \nabla_X Z + \mathrm{II}(X, Z) \right) - \widetilde{\nabla}_X \left( \nabla_Y Z + \mathrm{II}(Y, Z) \right) + \big( \nabla_{[X,Y]} Z + \mathrm{II}([X, Y], Z) \big) \,.
\end{aligned}$$

Employing now the linearity of the Levi-Civita connection and once again the Gauss formula,

$$\begin{aligned}
\widetilde{R}(X, Y)Z &= \widetilde{\nabla}_Y \nabla_X Z + \widetilde{\nabla}_Y \mathrm{II}(X, Z) - \widetilde{\nabla}_X \nabla_Y Z - \widetilde{\nabla}_X \mathrm{II}(Y, Z) + \nabla_{[X,Y]} Z + \mathrm{II}([X, Y], Z) \\
&= R(X, Y)Z + \mathrm{II}(Y, \nabla_X Z) + \widetilde{\nabla}_Y \mathrm{II}(X, Z) - \mathrm{II}(X, \nabla_Y Z) - \widetilde{\nabla}_X \mathrm{II}(Y, Z) \\
&\quad + \mathrm{II}([X, Y], Z).
\end{aligned}$$

Let $T \in \mathfrak{X}(M)$ be arbitrarily extended to $\widetilde{M}$. Then, since $T$ is tangent to $M$, we conclude from the definition of the Riemann curvatures $\widetilde{\mathrm{Rm}}$ and $\mathrm{Rm}$, respectively, that

$$\begin{aligned}
\widetilde{\mathrm{Rm}}(X, Y, Z, T) &= \langle \widetilde{R}(X, Y)Z, T \rangle \\
&= \langle R(X, Y)Z, T \rangle + \langle \widetilde{\nabla}_Y \mathrm{II}(X, Z), T \rangle - \langle \widetilde{\nabla}_X \mathrm{II}(Y, Z), T \rangle \\
&= \mathrm{Rm}(X, Y, Z, T) + \langle \widetilde{\nabla}_Y \mathrm{II}(X, Z), T \rangle - \langle \widetilde{\nabla}_X \mathrm{II}(Y, Z), T \rangle.
\end{aligned}$$

Finally, the proof follows applying the Weingarten equation to the last two terms above. $\square$

**Remark 3.27** *The Gauss equation relates the Riemann curvature* $\widetilde{\mathrm{Rm}}$ *of the ambient space* $\widetilde{M}$ *with the Riemann curvature* Rm *of the submanifold* $M$*. Note also that all the vector fields involved are tangent to the submanifold* $M$*.*

**Theorem 3.28 (Codazzi Equation)** *Let* $M$ *be a Riemannian submanifold of* $\widetilde{M}$*. Then, for every* $X, Y, Z \in \mathfrak{X}(M)$*,*

$$\left(\widetilde{R}(X,Y)Z\right)^{\perp} = \left(\overline{\nabla}_Y \mathrm{II}\right)(X,Z) - \left(\overline{\nabla}_X \mathrm{II}\right)(Y,Z)\,,$$

*where* $\overline{\nabla}$ *represents the Van Der Warden-Bortolotti connection defined as*

$$\overline{\nabla}_X \mathrm{II}(Y,Z) = \mathcal{D}_X \mathrm{II}(Y,Z) - \mathrm{II}(\nabla_X Y, Z) - \mathrm{II}(Y, \nabla_X Z)\,.$$

*Proof.* Following with the computation of $\widetilde{R}(X,Y)Z$ of the proof of the Gauss equation and employing Weingarten formula, it is easy to check that

$$\left(\widetilde{R}(X,Y)Z\right)^{\perp} = \mathrm{II}(Y, \nabla_X Z) + \mathcal{D}_Y \mathrm{II}(X,Z) - \mathrm{II}(X, \nabla_Y Z) - \mathcal{D}_X \mathrm{II}(Y,Z) + \mathrm{II}([X,Y],Z)\,,$$

since we are only considering the normal component to $M$. We then use in the last term that the Levi-Civita connection is symmetric to draw the conclusion. □

**Remark 3.29** *The Codazzi equation computes the Riemann curvature* $\widetilde{\mathrm{Rm}}$ *of the ambient space* $\widetilde{M}$ *when one of the vector fields involved is normal to* $M$ *(which one does not matter due to the symmetries of* $\widetilde{\mathrm{Rm}}$*).*

**Theorem 3.30 (Ricci Equation)** *Let* $M$ *be a Riemannian submanifold of* $\widetilde{M}$*. Then, for every* $X, Y \in \mathfrak{X}(M)$ *and* $\xi, \nu \in \mathfrak{N}(M)$*,*

$$\widetilde{\mathrm{Rm}}(X,Y,\xi,\nu) = \mathrm{Rm}^{\mathcal{D}}(X,Y,\xi,\nu) + \langle \mathrm{II}(X, \mathcal{A}_\xi Y), \nu\rangle - \langle \mathrm{II}(Y, \mathcal{A}_\xi X), \nu\rangle\,,$$

*where* $\mathrm{Rm}^{\mathcal{D}}$ *denotes the Riemann curvature of the normal bundle* $NM$ *(that is, with respect to the affine connection* $\mathcal{D}$*).*

*Proof.* Let $X, Y \in \mathfrak{X}(M)$ and $\xi, \nu \in \mathfrak{N}(M)$ and consider their arbitrary extensions to $\widetilde{M}$. From the definition of $\widetilde{R}$ and the Weingarten formula, we compute

$$\begin{aligned}
\widetilde{R}(X,Y)\xi &= \widetilde{\nabla}_Y \widetilde{\nabla}_X \xi - \widetilde{\nabla}_X \widetilde{\nabla}_Y \xi + \widetilde{\nabla}_{[X,Y]}\xi \\
&= -\widetilde{\nabla}_Y \mathcal{A}_\xi X + \widetilde{\nabla}_Y \mathcal{D}_X \xi + \widetilde{\nabla}_X \mathcal{A}_\xi Y - \widetilde{\nabla}_X \mathcal{D}_Y \xi - \mathcal{A}_\xi [X,Y] + \mathcal{D}_{[X,Y]}\xi\,.
\end{aligned}$$

Then, applying the Gauss formula to the terms in the first and third positions and the Weingarten formula to the second and fourth terms (and only considering the normal components), we obtain

$$\begin{aligned}
\widetilde{\mathrm{Rm}}(X,Y,\xi,\nu) &= -\langle \mathrm{II}(Y, \mathcal{A}_\xi X), \nu\rangle + \langle \mathcal{D}_Y \mathcal{D}_X \xi, \nu\rangle + \langle \mathrm{II}(X, \mathcal{A}_\xi Y), \nu\rangle \\
&\quad - \langle \mathcal{D}_X \mathcal{D}_Y \xi, \nu\rangle + \langle \mathcal{D}_{[X,Y]}\xi, \nu\rangle \\
&= \mathrm{Rm}^{\mathcal{D}}(X,Y,\xi,\nu) + \langle \mathrm{II}(X, \mathcal{A}_\xi Y), \nu\rangle - \langle \mathrm{II}(Y, \mathcal{A}_\xi X), \nu\rangle\,,
\end{aligned}$$

proving the result. □

**Remark 3.31** *The Ricci equation computes* $\widetilde{\mathrm{Rm}}$ *when two of the vector fields are normal to the submanifold.*

**Definition 3.32** *The equations of Gauss, Codazzi and Ricci are known as the* *fundamental equations.*

## 3.4 Surfaces in the Euclidean Space $\mathbb{R}^3$

(For the classical theory of surfaces, see [4] and/or [13]. An approach to surface theory via moving frames can be found in [6].)

**Definition 3.33** *A* *regular surface* $S$ *in* $\mathbb{R}^3$ *is a Riemannian submanifold of the Euclidean space* $\mathbb{R}^3$ *whose codimension is one.*

**Remark 3.34** *The notion of surface may be more general. It can be used to refer to Riemannian manifolds of diemnsion two. However, in our definition we are also requesting these* 2*-dimensional Riemannian manifolds to be isometrically immersed in* $\mathbb{R}^3$ *(cf. Remark 3.8).*

**Remark 3.35** *Let* $S$ *be a regular surface in* $\mathbb{R}^3$*. By definition,* $S$ *carries the induced metric from* $\mathbb{R}^3$*. If* $(u, v)$ *is a system of local coordinates, the coefficients of the metric are (in the notation of the classical theory of surfaces)*

$$E = \langle \partial_u, \partial_u \rangle \,, \qquad F = \langle \partial_u, \partial_v \rangle \,, \qquad G = \langle \partial_v, \partial_v \rangle \,.$$

*Since the codimension of* $S$ *in* $\mathbb{R}^3$ *is one, for every point* $p \in S$ *there exist two possible unit normal vectors. Locally, every surface is orientable and, hence, we can pick up the unit normal so that it gives the positive orientation. In the coordinates* $(u, v)$*, this unit normal is given by*

$$\xi = \frac{\partial_u \times \partial_v}{\sqrt{\langle \partial_u \times \partial_v, \partial_u \times \partial_v \rangle}} \,,$$

*where* $\times$ *denotes the usual vector product in* $\mathbb{R}^3$*.*

*We introduce the* *scalar second fundamental form* *by*

$$h(X, Y) = \langle \mathrm{II}(X, Y), \xi \rangle \,,$$

*for all* $X, Y \in \mathfrak{X}(S)$*. The coefficients of this form with respect to the coordinates* $(u, v)$ *are denoted (in the classical theory of surfaces) by*

$$e = h(\partial_u, \partial_u) \,, \qquad f = h(\partial_u, \partial_v) \,, \qquad g = h(\partial_v, \partial_v) \,.$$

**Theorem 3.36** *Let* $S$ *be a regular surface in* $\mathbb{R}^3$*. Then, for every* $X, Y \in \mathfrak{X}(S)$*,*

$$\nabla^{\mathbb{R}^3}_X Y = \nabla_X Y + h(X, Y)\xi \,,$$

*where* $\nabla^{\mathbb{R}^3}$ *is the Levi-Civita connection of* $\mathbb{R}^3$ *and* $\nabla$ *is the Levi-Civita connection of* $S$*.*

**Remark 3.37** *Theorem 3.36 is just a particular case of the Gauss formula (Theorem 3.20). Applying it to $\partial_u$ and $\partial_v$, we obtain the following expressions of the coefficients of the second fundamental form:*

$$e = \langle \partial^2_{uu}, \xi \rangle \,, \qquad f = \langle \partial^2_{uv}, \xi \rangle \,, \qquad g = \langle \partial^2_{vv}, \xi \rangle \,.$$

**Remark 3.38** *In the classical theory of surfaces, the Weingarten endomorphism $\mathcal{A}_\xi$ is usually denoted by $\mathcal{S}$ and called the shape operator, which we know it is a self-adjoint endomorphism (cf. Remark 3.25). For every $p \in S$, the shape operator at $p$ (denoted by $\mathcal{S}_p$) is diagonalizable and has two real eigenvalues $\kappa_1 \leq \kappa_2$.*

**Definition 3.39** *The real eigenvalues $\kappa_1 \leq \kappa_2$ of the shape operator $\mathcal{S}_p$ are called the principal curvatures of the surface $S$ at $p$.*

**Definition 3.40** *Let $S$ be a regular surface in $\mathbb{R}^3$. The Gaussian curvature of $S$ at a point $p \in S$ is the quantity*

$$K(p) = \det(\mathcal{S}_p) = \kappa_1 \kappa_2 \,,$$

*where $\mathcal{S}_p$ denotes the shape operator at $p$ and $\kappa_1 \leq \kappa_2$ are the principal curvatures of $S$ at $p$.*

*A surface $S$ in $\mathbb{R}^3$ is flat if the Gaussian curvature $K$ is zero at every point $p \in S$.*

**Remark 3.41** *A flat surface is not necessarily a part of a plane. For instance, a cylinder is flat but it is not contained in a plane.*

**Definition 3.42** *Let $S$ be a regular surface in $\mathbb{R}^3$. The mean curvature of $S$ at a point $p \in S$ is defined as*

$$H(p) = \frac{1}{2}\mathrm{trace}(\mathcal{S}_p) = \frac{1}{2}\left(\kappa_1 + \kappa_2\right),$$

*where $\mathcal{S}_p$ denotes the shape operator at $p$ and $\kappa_1 \leq \kappa_2$ are the principal curvatures of $S$ at $p$.*

*A surface $S$ in $\mathbb{R}^3$ is minimal if the mean curvature $H$ is zero at every point $p \in S$.*

**Remark 3.43** *There are several equivalent definitions of minimal surfaces. One such a definition is that minimal surfaces are those that locally minimize area. This approach to minimal surfaces as solutions of a variational problem is closely related to Chapter 5. The literature about the theory of minimal surfaces is vast. For example, in [15], more details about the topic can be found, while details about its direct extension to constant mean curvature surfaces appear in [11].*

**Theorem 3.44** *Let $S$ be a regular surface in $\mathbb{R}^3$. Then, for every $X \in \mathfrak{X}(S)$,*

$$\nabla^{\mathbb{R}^3}_X \xi = -\mathcal{S}X \,,$$

*where $\mathcal{S}$ denotes the shape operator and $\xi$ is the unit normal to $S$.*

**Remark 3.45** *The Weingarten equation provides us with an alternative way of interpreting the shape operator. Let $S$ be a regular surface in $\mathbb{R}^3$. We define the* Gauss map $\xi : S \subset \mathbb{R}^3 \longrightarrow \mathbb{S}^2$ *such that it associates to every point in the surface its unit normal vector. Then, the shape operator at $p \in S$ is*

$$\mathcal{S}_p = -d\xi_p \,,$$

*where $d\xi_p$ is the differential of the Gauss map at $p \in S$.*

**Remark 3.46** *From the Weingarten equation and the Weingarten formula, we obtain that the scalar second fundamental form $h$ can be expressed as*

$$h(X, Y) = \langle \mathcal{S}X, Y \rangle \,.$$

*It then follows that the matrix form of the shape operator $\mathcal{S}$ is given by $\mathcal{S} = g^{-1}h$ and, hence, the Gaussian curvature of $S$ at $p$ can be computed as*

$$K(p) = \frac{\det h}{\det g} = \frac{eg - f^2}{EG - F^2} \,.$$

**Theorem 3.47** *Let $S$ be a regular surface in $\mathbb{R}^3$ and consider the local coordinates $(u, v)$ in $S$. Then, the sectional curvature of $T_pS$ is given by*

$$K(T_pS) = \frac{eg - f^2}{EG - F^2} \,,$$

*where $E, G, F$ and $e, g, f$ are the coefficients of the metric and second fundamental form, respectively.*

**Remark 3.48** *Above result follows directly from the Gauss equation (Theorem 3.26) and shows that the Gaussian curvature of a surface $S$ in $\mathbb{R}^3$ is, precisely, the sectional curvature of $S$ viewed as a Riemannian manifold of dimension two. Furthermore, since the sectional curvature is defined intrinsically (observe that in Definition 2.35 there is no ambient space), above result also shows Gauss' Theorema Egregium.*

**Theorem 3.49 (Gauss' Theorema Egregium)** *The Gaussian curvature of a surface in $\mathbb{R}^3$ is invariant under local isometries.*

**Remark 3.50** *Roughly speaking, this means that the Gaussian curvature $K$ does not depend on how the surface might be immersed in $\mathbb{R}^3$. To the contrary, it can be determined entirely by measuring distances and angles on the surface itself. Indeed, in the local coordinates $(u, v)$ it can be computed employing, for instance, the following equation:*

$$\left(\Gamma^1_{11}\right)_v - \left(\Gamma^1_{12}\right)_u = \Gamma^1_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^1_{22} - F\,K \,,$$

*where $\Gamma^k_{ij}$, $i, j, k = 1, 2$, are the Christoffel symbols and $F = \langle \partial_u, \partial_v \rangle$. In the classical theory of surfaces above equation is known as the* Gauss equation for surfaces[7].

7. The Gauss equation for surfaces is not the Gauss equation of Theorem 3.26. To the contrary, it comes from the definition of the sectional curvature (Definition 2.35).

**Theorem 3.51 (Codazzi-Mainardi-Peterson Equations)** *Let $S$ be a regular surface in $\mathbb{R}^3$ and consider the local coordinates $(u,v)$ in $S$. Then the following equations hold,*

$$\begin{aligned} e_v - f_u &= e\Gamma^1_{21} + f\left(\Gamma^2_{21} - \Gamma^1_{11}\right) - g\Gamma^2_{11}\,, \\ f_v - g_u &= e\Gamma^1_{22} + f\left(\Gamma^2_{22} - \Gamma^1_{12}\right) - g\Gamma^2_{12}\,, \end{aligned}$$

*where $\Gamma^k_{ij}$, $i,j,k = 1,2$, are the Christoffel symbols and $e,f,g$ are the coefficients of the second fundamental form.*

**Remark 3.52** *The Codazzi-Mainardi-Peterson equations for surfaces follow directly from the Codazzi equation of Riemannian submanifolds (Theorem 3.28).*

**Remark 3.53** *In the classical theory of surfaces, the Gauss-Codazzi-Mainardi-Peterson equations (also known as the compatibility equations) are obtained from the identities*

$$\begin{aligned} \left(\partial^2_{uu}\right)_v - \left(\partial^2_{uv}\right)_u &= 0\,, \\ \left(\partial^2_{vv}\right)_u - \left(\partial^2_{uv}\right)_v &= 0\,, \\ \xi_{uv} - \xi_{vu} &= 0\,. \end{aligned}$$

*These identities follow from the symmetry of the second order mixed partial derivatives (see Clairaut-Schwarz's Theorem). The verification of these compatibility equations is necessary, and locally sufficient, to have a regular surface in $\mathbb{R}^3$ (see Bonnet's Theorem, Page 236 of [4]).*

*On the other hand, the Ricci equation in the case of surfaces in $\mathbb{R}^3$ is an identity since the codimension of $S$ in $\mathbb{R}^3$ is one.*

## 3.5 Exercises

1. Let $M$ be a Riemannian submanifold of $\widetilde{M}$ and denote by $\widetilde{\nabla}$ the Levi-Civita connection of $\widetilde{M}$.

   i) Check that the tangential projection of the covariant derivative $\widetilde{\nabla}_X Y$ to $M$, for all $X, Y \in \mathfrak{X}(M)$ defines an affine connection $\nabla$ over $M$.

   ii) Prove that this affine connection $\nabla$ defined over $M$ is, precisely, the Levi-Civita connection of $M$.

   This proves Proposition 3.17.

2. Prove Weingarten equation (Theorem 3.24).

3. Explain why $EG - F^2 > 0$ at every point of a surface in $\mathbb{R}^3$.

4. Consider the surface of revolution $S$ in $\mathbb{R}^3$ obtained by rotating the curve $\gamma$ given by $\gamma(s) = (f(s), 0, g(s))$, $f(s) > 0$, (assume that $s$ is the arc length parameter of the curve, that is, $f'(s)^2 + g'(s)^2 = 1$ holds) around the $z$-axis. Compute the Gaussian and mean curvatures of $S$.

5. Consider the surface $S$ in $\mathbb{R}^3$ given, locally, by the graph $z = f(x, y)$. Compute the Gaussian and mean curvatures of $S$.

6. * Let $S$ be a regular surface in $\mathbb{R}^3$ with Gaussian curvature $K$ and mean curvature $H$.

   i) Obtain a explicit expression of the principal curvatures $\kappa_1 \leq \kappa_2$ of a surface $S$ at a point $p \in S$ in terms of the Gaussian and mean curvatures.

   ii) Prove that
   $$H^2 - K \geq 0\,,$$
   holds at every point $p \in S$.

   iii) Show that minimal surfaces have non-positive Gaussian curvature.

   iv) The points $p \in S$ such that $H^2(p) = K(p)$ are called umbilical points. Show that the principal curvatures at a point are equal if and only if the point is umbilical.

   v) A surface $S$ is said to be totally umbilical if all points of $S$ are umbilical. Show that if $S$ is totally umbilical, then $S$ is a part of a plane (if $H^2 = K = 0$) or a part of a sphere (if $H^2 = K > 0$).

7. * Let $\xi : S \longrightarrow \mathbb{S}^2 \subset \mathbb{R}^3$ be the Gauss map of the regular surface $S$ in $\mathbb{R}^3$. The third fundamental form of $S$ is defined by
$$\mathrm{III} = \langle d\xi, d\xi \rangle\,.$$

   i) Show that
   $$\mathrm{III} = 2Hh - Kg\,,$$
   where $H$ and $K$ are the mean and Gaussian curvatures of $S$, respectively, and $h$ is the scalar second fundamental form of $S$.

   ii) Compute the characteristic polynomial of the shape operator $\mathcal{S}$.

   iii) Apply the Cayley-Hamilton's Theorem to the above polynomial and the shape operator matrix.

8. * Let $S$ be a regular surface in $\mathbb{R}^3$ and $\phi : S \longrightarrow \mathbb{R}^3$ the isometric immersion. A system of local coordinates $(u, v)$ in $S$ is called an isothermal coordinate system if
$$\langle \partial_u, \partial_u \rangle = \langle \partial_v, \partial_v \rangle = \mu^2\,, \qquad \langle \partial_u, \partial_v \rangle = 0\,,$$
hold.

   i) Show that for a isothermal coordinate system,
   $$\partial^2_{uu} + \partial^2_{vv} = 2\mu^2 H\, \xi\,,$$
   where $H$ is the mean curvature of $S$ and $\xi$ is the unit normal.

ii) Show that if $x^{-1}$ is the local parameterization of $S$ given by the isothermal coordinate system $(u, v)$, then

$$\Delta\left(\phi \circ x^{-1}\right) = 2\mu^2 H\, \xi\,,$$

and, hence, $S$ is minimal if and only if the coordinate functions $\phi \circ x^{-1}$ are harmonic. (Recall that an harmonic function $f$ is a solution of the Laplace equation $\Delta f = 0$.)

The heat equation in the plane is the partial differential equation $\partial_t f = \Delta f$. In the plane, the Laplacian vanishes when applied to isothermal coordinates and, hence, these coordinates are a steady solution to the heat equation. In other words, the temperature remains constant along time and so the name isothermal.

9. Obtain alternative expressions of the Gauss equation for surfaces in $\mathbb{R}^3$ (cf. Remark 3.50).

10. * Let $S$ be a regular surface in $\mathbb{R}^3$. The surface $S$ is isoparametric if its principal curvatures are constant functions, that is, for every $p \in S$ the values $\kappa_i(p)$, $i = 1, 2$, are the same.

    i) Show that $S$ is isoparametric if all its parallel surfaces $S_r = \{p + r\xi \,|\, p \in S\}$, for $|r| < \epsilon$ small enough, have constant mean curvature. Here, $\xi$ denotes the unit normal to $S$.

    ii) Prove that isoparametric surfaces in $\mathbb{R}^3$ are parts of either totally umbilical surfaces or spherical cylinders.

# 4 Geodesics in Riemannian Manifolds

(For more details, see Chapter 3 of [5] and/or Chapter 6 of [9].)

## 4.1 Curves in Riemannian Manifolds

**Definition 4.1** *A smooth immersed curve in a Riemannian manifold is a Riemannian submanifold of dimension one.*

**Remark 4.2** *Smooth immersed curves in $M$ are denoted by $\gamma : J \subseteq \mathbb{R} \longrightarrow M$ where $J$ will be assumed to be the maximal interval of definition and, unless explicitly mentioned, we will not distinguish between the map $\gamma$ and its trajectory (or, trace) $\gamma(J) \subseteq M$. Both will be referred to as curves.*

**Remark 4.3** *In the classical theory, curves are defined in the following equivalent way. A smooth parameterized curve in a Riemannian manifold $M$ is a smooth map $\gamma : J \subseteq \mathbb{R} \longrightarrow M$. These curves may have cusps which, for our purposes, are undesired. To avoid them, the notion of regular curves is introduced. A smooth parameterized curve is said to be regular if $\dot{\gamma}(t) \neq 0$ for every $t \in J$. Here, the upper dot denotes the derivative with respect to the parameter $t \in J$.*

**Definition 4.4** *Let $\gamma : J = (a, b) \subseteq \mathbb{R} \longrightarrow M$ be a smooth immersed curve in a Riemannian manifold $M$. The length of $\gamma$ is defined by*

$$L(\gamma) = \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} \, dt \,,$$

*where $\dot{\gamma} = d\gamma/dt$ and $t \in J = (a, b)$ is the parameter of $\gamma$.*

**Definition 4.5** *Let $M$ be a Riemannian manifold and $p, q \in M$ be two points. The Riemann distance between $p$ and $q$, $d(p, q)$, is the infimum of the lengths of all smooth immersed curves joining $p$ to $q$. That is,*

$$d(p, q) = \inf_{\gamma} L(\gamma) \,,$$

*among all $\gamma : J = (a, b) \subseteq \mathbb{R} \longrightarrow M$ such that $\gamma(a) = p$ and $\gamma(b) = q$.*

**Remark 4.6** *Since $M$ is connected (recall our assumption of Remark 1.4), every pair of points in $M$ can be joined by a smooth immersed curve and, hence, the Riemann distance is well defined.*

**Theorem 4.7** *Let $M$ be a Riemannian manifold. Endowed with the Riemann distance, $M$ becomes a metric space whose induced topology is the same as the one that $M$ carries as a topological manifold.*

**Remark 4.8** *So far, we have used an arbitrary parameter $t \in J$ for smooth immersed curves. However, these curves admit a reparameterization by a natural parameter that makes their speed $V(t) = \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}$ to be constant one, namely, the arc length parameter.*

**Definition 4.9** *A smooth immersed curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ is said to be parameterized by the arc length if $\langle \gamma'(s), \gamma'(s) \rangle = 1$ for all $s \in I$. The parameter $s \in I$ is called the arc length parameter (or, natural parameter). Curves parameterized by the arc length may be referred to as unit speed curves.*

**Remark 4.10** *To distinguish from an arbitrary parameter $t \in J = (a, b)$, the derivative with respect to the arc length parameter $s \in I = (0, L)$ is denoted by $(\,)'$.*

**Proposition 4.11** *Let $\gamma : J = (a, b) \subseteq \mathbb{R} \longrightarrow M$ be a smooth immersed curve. Define the function $\mathfrak{s} : J \longrightarrow I$ by*

$$\mathfrak{s}(t) = \int_a^t \sqrt{\langle \dot{\gamma}(u), \dot{\gamma}(u) \rangle}\, du\,,$$

*and denote by $\mathtt{t}$ to its inverse. Then, the curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ given by $\gamma(s) = \gamma(\mathtt{t}(s))$ is a smooth curve parameterized by the arc length.*

**Remark 4.12** *In other words, $\gamma(s) = \gamma(\mathtt{t}(s))$ is a reparameterization of $\gamma(t)$ by arc length. Both curves have the same trajectory (or, trace).*

**Definition 4.13** *Let $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ be a smooth immersed curve parameterized by the arc length $s \in I$. The unit tangent vector field along $\gamma$ is $T(s) = \gamma'(s)$. The curvature $\kappa(s)$ of $\gamma$ is the function*

$$\kappa(s) = \sqrt{\langle \nabla_T T(s), \nabla_T T(s) \rangle}\,,$$

*where $\nabla$ is the Levi-Civita connection of $M$.*

**Remark 4.14** *Our definition for the curvature of a curve implies that $\kappa(s)$ is a non-negative smooth function. However, in the particular case of a Riemannian manifold of dimension two, it is possible to give the curvature a sign and define what is known as the signed curvature $\mathtt{k}$. For that we define the (oriented) unit normal $\mathtt{N}(s)$ by requiring that $\{T(s), \mathtt{N}(s)\}$ is a positively oriented basis of the tangent plane for all $s \in I$. The signed curvature $\mathtt{k}(s)$ is then defined as*

$$\nabla_T T(s) = \mathtt{k}(s)\mathtt{N}(s)\,,$$

*and might be either positive, negative or both. In this context, the points $\gamma(s_o)$, $s_o \in I$, such that $\mathtt{k}(s_o) = 0$ are called inflection points.*

**Definition 4.15** *An arc length parameterized smooth immersed curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M^n$ is called a Frenet curve of rank one if its curvature $\kappa(s)$ is identically zero. It is said a Frenet curve of rank m, $2 \leq m \leq n$, if $m$ is the largest integer for which there exists an orthonormal frame* *defined along $\gamma$, $\{e_1(s) = T(s), e_2(s), ..., e_m(s)\}$ and non-negative smooth functions defined on $\gamma$, $\kappa_i(s)$, $1 \leq i \leq m-1$, called Frenet curvatures, such that,*

$$\begin{aligned} \nabla_T e_1(s) &= \kappa_1(s) e_2(s) \,, \\ \nabla_T e_h(s) &= -\kappa_{h-1}(s) e_{h-1}(s) + \kappa_h(s) e_{h+1}(s) \,, \qquad h = 2, ..., m-1 \,, \\ \nabla_T e_m(s) &= -\kappa_{m-1}(s) e_{m-1}(s) \,. \end{aligned}$$

*These equations are known as the Frenet-Serret equations.*

**Remark 4.16** *For a Frenet curve of rank $m < n$, the Frenet curvatures of index larger than $m-1$ are considered to be zero. The first Frenet curvature $\kappa_1(s) \equiv \kappa(s)$ is, simply, the curvature of $\gamma$. The second Frenet curvature $\kappa_2(s) \equiv \tau(s)$ is usually called the torsion of $\gamma$. A curve with vanishing torsion, $\tau(s) = 0$ for all $s \in I$ (in other words, a Frenet curve of rank two), is called a planar curve.*

**Definition 4.17** *A smooth immersed curve whose Frenet curvatures are all constant functions is called a Frenet helix.*

**Remark 4.18** *If $M$ is a Riemannian manifold of dimension three, the Frenet-Serret equations along $\gamma$ are usually expressed as*

$$\begin{aligned} \nabla_T T(s) &= \kappa(s) N(s) \,, \\ \nabla_T N(s) &= -\kappa(s) T(s) + \tau(s) B(s) \,, \\ \nabla_T B(s) &= -\tau(s) N(s) \,, \end{aligned}$$

*where $N(s)$ is the unit normal[8] vector field along $\gamma$ and $B(s)$ is the unit binormal. Even if the curve $\gamma$ is planar ($\tau(s) = 0$ for all $s \in I$ or, equivalently, the rank of $\gamma$ is two), the binormal $B(s)$ is still well defined as $B = T \times N$, where $\times$ is the usual vector product of $\mathbb{R}^3$.*

## 4.2 Geodesics

**Definition 4.19** *A smooth immersed curve $\gamma : J \subseteq \mathbb{R} \longrightarrow M$ is a geodesic if for all $t \in J$,*

$$\nabla_{\dot\gamma} \dot\gamma(t) = 0 \,,$$

*where $\nabla$ is the Levi-Civita connection of $M$ and $\dot\gamma$ is extended arbitrarily to a neighborhood of $\gamma(J)$.*

**Remark 4.20** *A vector field $\mathcal{V}$ along a curve $\gamma$ is said parallel along $\gamma$ if $\nabla_{\dot\gamma} \mathcal{V} = 0$. Therefore, a geodesic is a curve whose velocity $\dot\gamma(t)$ is parallel along $\gamma$. The vector field $\nabla_{\dot\gamma}\dot\gamma(t)$ along the curve $\gamma$ is sometimes referred to as the acceleration of $\gamma$. With this notion, a geodesic is a curve with vanishing acceleration.*

8. Do not confuse with the oriented unit normal `N` defined in Remark 4.14.

**Proposition 4.21** *Let $\gamma : J \subseteq \mathbb{R} \longrightarrow M$ be a geodesic. Then, the speed $V(t) = \sqrt{\langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle}$ of $\gamma$ is constant. That is, geodesics are parameterized with constant speed.*

*Proof.* To prove the result is enough to check that the speed of a geodesic $\gamma$ is constant. For this, we will see that its derivative (more precisely, the derivative of its square) is zero. Denote by

$$V^2(t) = \langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle \,,$$

the square of the speed. Then,

$$\frac{d}{dt}\left(V^2(t)\right) = \frac{d}{dt}\langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle = 2\langle\nabla_{\dot{\gamma}}\dot{\gamma}(t), \dot{\gamma}(t)\rangle = 0 \,,$$

because $\nabla_{\dot{\gamma}}\dot{\gamma}(t) = 0$ holds for geodesics. □

**Remark 4.22** *The previous result shows that the choice of parameterization is essential to have geodesics. An arbitrary reparameterization of a geodesic curve may not be a geodesic, although their trajectories are the same.*

**Example 4.23** *Constant speed parameterized straight lines in the Euclidean plane $\mathbb{R}^2$ are geodesics. However, the straight line parameterized by $\gamma(t) = (t^3, 0)$, $t \in \mathbb{R}$, is not a geodesic.*

**Proposition 4.24** *Let $\gamma : J \subseteq \mathbb{R} \longrightarrow M$ be a smooth immersed curve parameterized with constant speed. Then, $\gamma$ is a geodesic if and only if its curvature $\kappa$ vanishes identically.*

*Proof.* Since $\gamma$ has constant speed, $V^2(t) = \langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle = c^2$, for some positive constant $c \in \mathbb{R}^+$. It then follows that (cf. Proposition 4.11)

$$\mathfrak{s}(t) = \int_a^t \sqrt{\langle\dot{\gamma}(u), \dot{\gamma}(u)\rangle}\, du = \int_a^t c\, du = c(t-a) \,,$$

and its inverse is $\mathfrak{t}(s) = s/c + a$. In other words, *the parameter $t \in J$ of a geodesic is an affine function of the arc length parameter $s \in I$.*

We then reparameterize $\gamma$ by the arc length as $\gamma(s) = \gamma(\mathfrak{t}(s)) = \gamma(s/c + a)$ and obtain, from the chain rule, that

$$T(s) = \gamma'(s) = \frac{1}{c}\dot{\gamma}(t) \,.$$

Finally, from the definition of the curvature, this relation between the velocity vector fields, and the properties of an affine connection (see Definition 2.12),

$$\kappa^2(s) = \langle\nabla_T T(s), \nabla_T T(s)\rangle = \langle\nabla_{\dot{\gamma}/c}\dot{\gamma}(t)/c, \nabla_{\dot{\gamma}/c}\dot{\gamma}(t)/c\rangle = \frac{1}{c^4}\langle\nabla_{\dot{\gamma}}\dot{\gamma}(t), \nabla_{\dot{\gamma}}\dot{\gamma}(t)\rangle \,.$$

The equivalence of the statement follows directly from this. □

**Remark 4.25** *In particular, if $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ is an arc length parameterized smooth immersed curve, the notions of geodesic and Frenet curve of rank one are equivalent.*

**Theorem 4.26 (Local Equations for Geodesics)** *Let $(U, x)$ be a local chart for a Riemannian manifold $M$ and consider a smooth immersed curve $\gamma : J \subseteq \mathbb{R} \longrightarrow M$ such that $\gamma(J) \subset U$. Then, $\gamma$ is a geodesic if and only if locally its coordinate functions $x_k(t)$, $k = 1, ..., n$, satisfy the system of second order ordinary differential equations*

$$\ddot{x}_k(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} \dot{x}_i(t)\dot{x}_j(t)\Gamma_{ij}^k\left(x(t)\right) = 0\,,$$

*where $\Gamma_{ij}^k$ are the Christoffel symbols of the Levi-Civita connection with respect to $(U, x)$.*

*Proof.* By definition, the curve $\gamma$ is a geodesic if and only if $\nabla_{\dot{\gamma}}\dot{\gamma}(t) = 0$ for every $t \in J$. In the local chart $(U, x)$ we write

$$\dot{\gamma}(t) = \sum_{i=1}^{n} \dot{x}_i(t)\partial_{x_i}\,,$$

where $\{\partial_{x_i}\}_{i=1}^n$ is the local frame of the tangent bundle of $M$. It then follows from the properties of the affine connection $\nabla$ (see Definition 2.12) that

$$\begin{aligned}
\nabla_{\dot{\gamma}}\dot{\gamma}(t) &= \nabla_{\dot{\gamma}}\left(\sum_{i=1}^{n} \dot{x}_i(t)\partial_{x_i}\right) = \sum_{i=1}^{n} \nabla_{\dot{\gamma}}\left(\dot{x}_i(t)\partial_{x_i}\right) = \sum_{i=1}^{n}\left(\ddot{x}_i(t)\partial_{x_i} + \dot{x}_i(t)\nabla_{\dot{\gamma}}\partial_{x_i}\right) \\
&= \sum_{i=1}^{n} \ddot{x}_i(t)\partial_{x_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} \dot{x}_i(t)\dot{x}_j(t)\nabla_{\partial_{x_j}}\partial_{x_i} \\
&= \sum_{i=1}^{n} \ddot{x}_i(t)\partial_{x_i} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \dot{x}_i(t)\dot{x}_j(t)\Gamma_{ij}^k(x(t))\partial_{x_k} \\
&= \sum_{k=1}^{n}\left(\ddot{x}_k(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} \dot{x}_i(t)\dot{x}_j(t)\Gamma_{ij}^k(x(t))\right)\partial_{x_k}\,,
\end{aligned}$$

where in the last equality we have reorganized the terms of the first sum. Consequently, $\gamma$ is a geodesic if and only if all the components of the vector field above are zero. □

**Example 4.27** *Compute all the geodesics of the Euclidean space $\mathbb{R}^n$.*

**Theorem 4.28 (Existence and Uniqueness of Geodesics)** *Let $M$ be a Riemannian manifold. For every $p \in M$ and $X_p \in T_pM$, there exists an open interval $J \subseteq \mathbb{R}$ and a unique geodesic $\gamma : J \subseteq \mathbb{R} \longrightarrow M$ such that $\gamma(t_o) = p$ and $\dot{\gamma}(t_o) = X_p$, for some $t_o \in J$.*

*Proof.* From the local equations for geodesics, we know that $\gamma$ is a geodesic if and only if its coordinate functions $x_k(t)$, $k = 1, ..., n$ satisfy

$$\ddot{x}_k(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} \dot{x}_i(t)\dot{x}_j(t)\Gamma_{ij}^k\left(x(t)\right) = 0\,.$$

We transform these second order differential equations into a system of first order equations by introducing the variables $v_k(t) = \dot{x}_k(t)$, $k = 1, ..., n$. With these new variables, above second order equations read

$$\begin{cases} \dot{x}_k(t) = v_k(t) \\ \dot{v}_k(t) = -\sum_{i=1}^{n}\sum_{j=1}^{n} v_i(t)v_j(t)\Gamma_{ij}^k(x(t)) \end{cases} .$$

Together with the initial conditions specified in the statement, these are initial value problems whose local existence and uniqueness follows from the standard theory of systems of first order ordinary differential equations. □

**Remark 4.29** *Observe that shifting the parameter $t \in J$, if necessary, we may assume that $t_o = 0$. For every $X = (p, X_p) \in TM$ we will denote by $\gamma_X$ the unique geodesic with initial conditions $\gamma_X(0) = p$ and $\dot{\gamma}_X(0) = X_p$. We will also use the notation $J_X$ for the maximal interval of definition of $\gamma_X$.*

## 4.3 Exponential Map

**Definition 4.30** *Let $M$ be a Riemannian manifold and consider $U \subseteq TM$ defined by*

$$U = \{X \in TM \,|\, [0,1] \subseteq J_X\} .$$

*The exponential map* $\exp : U \longrightarrow M$ *is given by*

$$\exp(X) = \gamma_X(1) .$$

*For every $p \in M$, we denote by $U_p = U \cap T_pM$ and refer to $\exp_p : U_p \longrightarrow M$ as the (restricted) exponential map.*

**Proposition 4.31** *Let* $\exp$ *be the exponential map of a Riemannian manifold $M$. Then, the followings hold:*

1. *The subset $U$ is open in $TM$ and it contains the zero section $(p, 0)$. Moreover, for every $p \in M$, $U_p$ is star-shaped with respect to the origin.*
2. *For every $X \in TM$, the geodesic $\gamma_X$ is given by*
$$\gamma_X(t) = \exp(tX) ,$$
*for every $t \in J_X$.*
3. *The exponential map is a smooth map.*
4. *Rescaling Lemma. For every $X \in TM$ and $c \in \mathbb{R}$,*
$$\gamma_{cX}(t) = \gamma_X(c\,t) ,$$
*as long as $t \in J_{cX}$ and $c\,t \in J_X$.*

**Remark 4.32** *From Point 3 above, we have that the exponential map* $\exp_p$ *is smooth. However, even if* $\exp_p$ *is defined on the entire tangent space* $T_pM$*, it may not be a diffeomorphism. Nonetheless, its differential at the origin is the identity and, hence, by the Inverse Function Theorem,* $\exp_p$ *is a local diffeomorphism. The* injectivity radius *of* $M$ *at* $p \in M$ *is the largest radius of a ball that can be diffeomorphically map via* $\exp_p$.

**Definition 4.33** *Let* $M$ *be a Riemannian manifold* $M$ *and* $p \in M$*. For every two-dimensional linear subspace* $\pi \subseteq T_pM$*, the* planar section *determined by* $\pi$ *is*

$$S_\pi = \exp_p(\pi \cap V_p),$$

*where* $V_p \subseteq T_pM$ *is a neighborhood of the zero where* $\exp_p$ *is a diffeomorphism.*

**Remark 4.34** *The planar section* $S_\pi$ *is a Riemannian submanifold of* $M$ *of dimension two which contains the point* $p \in M$ *and is composed by the geodesics starting at* $p \in M$ *and with initial velocity in the subspace* $\pi \subseteq T_pM$*. Extending in a standard way the notion of Gaussian curvature (cf. Definition 3.40) to surfaces in* $M$*, not necessarily in* $\mathbb{R}^3$*, we obtain that the sectional curvature of* $\pi$ *at* $p \in M$ *(defined in* $M$*) is, precisely, the Gaussian curvature of the planar section.*

**Lemma 4.35 (Gauss' Lemma)** *Let* $M$ *be a Riemannian manifold and* $p \in M$*. The image of a sphere of sufficiently small radius via the exponential map* $\exp_p$ *is perpendicular to all geodesics starting at* $p \in M$*.*

**Definition 4.36** *A Riemannian manifold is said to be* geodesically complete *if the maximal interval of definition of every geodesic is the entire real line* $\mathbb{R}$*.*

**Theorem 4.37 (Hopf-Rinow)** *A Riemannian manifold is geodesically complete if and only if it is complete as a metric space.*

**Corollary 4.38** *Let* $M$ *be a Riemannian manifold. If there exists a point* $p \in M$ *such that the exponential map* $\exp_p$ *is defined in all the tangent space* $T_pM$*, then* $M$ *is complete.*

**Remark 4.39** *A compact Riemannian manifold* $M$ *is complete. Hence, every geodesic can be defined in the entire real line* $\mathbb{R}$*.*

**Remark 4.40** *A Riemannian manifold* $M$ *is complete if and only if every two points of* $M$ *can be joined by a minimizing geodesic. A minimizing geodesics is, roughly speaking, a geodesic that minimizes the length between the two points. We will study this concept with more detail in the next chapter.*

## 4.4 Exercises

1. Show that the notion of a smooth parameterized regular curve in a Riemannian manifold (cf. Remark 4.3) is equivalent to that of a smooth immersed curve (cf. Definition 4.1).

2. Show that every smooth immersed curve in a Riemannian manifold admits a reparameterization by the arc length. (Proposition 4.11.)

3. Let $\gamma : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$ be a smooth immersed curve in the Euclidean space $\mathbb{R}^3$ parameterized by an arbitrary parameter $t \in J$.

   i) Show that the curvature $\kappa$ of $\gamma$ is given by

   $$\kappa(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3} \,,$$

   where $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm of a vector $u \in \mathbb{R}^3$.

   ii) Show that the torsion $\tau$ of $\gamma$ is given by

   $$\tau(t) = \frac{\langle \dot{\gamma}(t) \times \ddot{\gamma}(t), \dddot{\gamma}(t)\rangle}{\langle \dot{\gamma}(t) \times \ddot{\gamma}(t), \dot{\gamma}(t) \times \ddot{\gamma}(t)\rangle} \,.$$

   iii) Assume now that $\gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$ is parameterized by the arc length $s \in I$. Show that the torsion $\tau$ of $\gamma$ is then

   $$\tau(s) = \frac{\langle \gamma'(s) \times \gamma''(s), \gamma'''(s)\rangle}{\kappa^2(s)} \,,$$

   where $\kappa(s)$ is the curvature of $\gamma$.

   iv) Let $\gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$ be an arc length parameterized planar curve (that is, $\tau = 0$ identically). Then, the curve $\gamma$ may be understood as $\gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$. Consider a fixed direction $v$ in $\mathbb{R}^2$ and define $\theta(s)$ as the angle between $\gamma'(s)$ and $v$. Show that

   $$\kappa(s) = |\theta'(s)| \,,$$

   holds.

4. Show that the geodesics of the hyperbolic plane $\mathbb{H}^2$ in the upper half-plane model, that is, $\mathcal{H} = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0\}$ endowed with the metric

   $$g = \frac{1}{y^2}\left(dx^2 + dy^2\right) ,$$

   are circles and straight lines (properly parameterized) which meet the $x$-axis at right angles.

5. * Prove that the geodesics of a sphere $\mathbb{S}^n$ are parts of great circles parameterized with constant speed.

6. * Let $M$ be a Riemannian submanifold of $\widetilde{M}$ and consider a smooth immersed curve $\gamma : J \subseteq \mathbb{R} \longrightarrow M$ parameterized with constant speed.

   i) Show that $\gamma$ is a geodesic if and only if the acceleration vector field along $\gamma$ is normal to $M$.

   ii) The Riemannian submanifold $M$ is called <u>totally geodesic</u> if any geodesic of $M$ is also a geodesic of $\widetilde{M}$. Show that a submanifold $M$ is totally geodesic if and only if its second fundamental form is identically zero.

   iii) Classify all totally geodesic submanifolds of the Euclidean space $\mathbb{R}^n$.

   iv) Prove that the <u>equator</u> $\mathbb{S}^{n-1}$ is a totally geodesic submanifold of $\mathbb{S}^n$.

7. * Consider the surface of revolution $S$ in $\mathbb{R}^3$ given locally by

$$x^{-1}(s, \theta) = (f(s)\cos\theta, f(s)\sin\theta, g(s)) ,$$

   where $f(s) > 0$ and $f'(s)^2 + g'(s)^2 = 1$.

   i) Show that all the <u>meridians</u> $\theta = \theta_o$ constant are geodesics.

   ii) Show that a <u>parallel</u> $s = s_o$ constant parameterized by the arc length is a geodesic if and only if $s_o$ is a critical point of the function $f$.

   iii) Prove <u>Clairaut's relation</u>: If $\gamma(t)$ is a geodesic of a surface of revolution, then

$$r(t)\cos\psi(t) = c \in \mathbb{R} ,$$

   where $r(t)$ represents the radius of the parallel intersecting $\gamma$ at $\gamma(t)$ and $\psi(t)$ is the angle made by $\gamma$ and that parallel.

8. * Let $M$ be a Riemannian manifold and $f \in \mathcal{C}^\infty(M)$ such that $\langle \operatorname{grad} f, \operatorname{grad} f \rangle = 1$ at every point $p \in M$ (for the definition of the gradient vector field see Exercise 2 of Section 2.5). Show that the integral curves of $\operatorname{grad} f$ are geodesics.

9. * Let $S$ be a regular surface in $\mathbb{R}^3$ and consider a smooth immersed curve $\gamma : I \subseteq \mathbb{R} \longrightarrow S \subset \mathbb{R}^3$ parameterized by the arc length $s \in I$. The <u>normal curvature</u> $\kappa_n$ of $\gamma$ is defined by

$$\kappa_n = \langle \nabla^{\mathbb{R}^3}_T T, \xi \rangle ,$$

   where $T = \gamma'$ is the unit tangent vector field along $\gamma$. A curve whose normal curvature is identically zero is an <u>asymptotic curve</u>.

   i) Denote by $\theta \in [-\pi, \pi]$ the oriented angle between the unit normal $N$ along $\gamma$ (as a curve in $\mathbb{R}^3$) and the normal $\xi$ to the surface. Show that

$$\kappa_n = \kappa \cos\theta ,$$

   where $\kappa$ is the curvature of $\gamma$ as a curve in $\mathbb{R}^3$.

ii) Define the conormal $n(s)$ as the vector field along $\gamma$ given by $n = T \times \xi$, where $\times$ denotes the usual vector product in $\mathbb{R}^3$. The positively oriented frame $\{n, T, \xi\}$ along $\gamma$ is called the Darboux frame. The geodesic curvature[9] $\kappa_g$ of $\gamma$ is

$$\kappa_g = \langle \nabla_T^{\mathbb{R}^3} T, n \rangle \,.$$

Show that $\kappa_g = \kappa \sin\theta$ (where $\theta$ is as in the previous case) and, hence, $\kappa^2 = \kappa_g^2 + \kappa_n^2$ holds.

iii) Use the Gauss formula for surfaces (Theorem 3.36) to obtain $\kappa^2 = \kappa_g^2 + \kappa_n^2$, in a different way. Deduce that

$$\kappa_n = h(\gamma', \gamma') \,,$$

where $h$ is the scalar second fundamental form of $S$, and that $\kappa_g$ is, in absolute value, the curvature of $\gamma$ as a curve in $S$.

iv) Consider an orthonormal frame $\{e_1, e_2\}$ of the tangent bundle to $S$. Show that for a unit vector field $X$, the normal curvature in the direction of $X$ is given by

$$\kappa_n = \kappa_1 \cos^2\psi + \kappa_2 \sin^2\psi \,,$$

where $\psi$ is the angle between $e_1$ and $X$, and $\kappa_1 \leq \kappa_2$ are the principal curvatures of $S$. This is known as the Euler's formula.

v) Show that the maximum and minimum normal curvatures are the principal curvatures.

vi) Denote by $\kappa_n(\psi)$ the normal curvature in the direction $X$ making an angle $\psi$ with a fixed direction. Prove that the mean curvature $H$ of $S$ is given by

$$H = \frac{1}{\pi} \int_0^\pi \kappa_n(\psi)\, d\psi \,.$$

10. * Let $\gamma : I \subseteq \mathbb{R} \longrightarrow S \subset \mathbb{R}^3$ be an arc length parameterized smooth immersed curve in a regular surface $S$ in $\mathbb{R}^3$ and denote by $n(s)$ the conormal vector field along $\gamma$ (see the previous exercise for its definition). The geodesic torsion $\tau_g$ of $\gamma$ is defined by

$$\tau_g = \langle \nabla_T^{\mathbb{R}^3} n, \xi \rangle \,.$$

i) Let $\theta \in [-\pi, \pi]$ be the oriented angle between the unit normal $N$ along $\gamma$ (as a curve in $\mathbb{R}^3$) and $\xi$. Prove that

$$\tau_g = \theta' - \tau \,,$$

where $\tau$ is the torsion of $\gamma$ (as a curve in $\mathbb{R}^3$).

9. Observe that most authors define $\kappa_g$ with the opposite sign.

ii) A principal direction is such that the normal curvature $\kappa_n$ of $\gamma$ in that direction is one of the principal curvatures $\kappa_i$, $i = 1, 2$, of $S$. Denote by $\{e_1, e_2\}$ the orthonormal frame of the tangent bundle giving the principal directions. Show that

$$\tau_g = (\kappa_1 - \kappa_2) \cos\psi \sin\psi \,,$$

where $\psi$ is the angle between $T$ and $e_1$.

iii) A line of curvature is a curve whose tangent is always a principal direction. Prove that lines of curvature are characterized by having identically zero geodesic torsion.

# 5 Introduction to Calculus of Variations

(For more details about the variation formulas of the length and energy functionals, see Chapter 9 of [5] and/or Chapters 6 and 10 of [9]. For a deeper understanding of the Calculus of Variations, see [7].)

## 5.1 Length and Energy Functionals

**Definition 5.1** *Let $\gamma : J = (a,b) \subseteq \mathbb{R} \longrightarrow M$ be a smooth immersed curve in a Riemannian manifold $M$. The length of $\gamma$ is*

$$L(\gamma) = \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}\, dt\,,$$

*while the energy of $\gamma$ is defined by*

$$E(\gamma) = \int_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle\, dt\,.$$

**Remark 5.2** *The reason why we introduce the energy and will work with it instead of with the length is based on two facts: on one hand, we avoid working with square roots, which make computations tedious, and on the other hand, we will obtain that critical points are parameterized with constant speed, which we recall here that it is essential to have geodesics (Proposition 4.21).*

**Proposition 5.3** *Let $\gamma : J = (a,b) \subseteq \mathbb{R} \longrightarrow M$ be a smooth immersed curve. Then,*

$$L^2(\gamma) \leq (b-a)E(\gamma)\,,$$

*holds between the length $L$ and the energy $E$ of $\gamma$. Moreover, equality holds if and only if $\gamma$ has constant speed $V(t) = \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}$.*

*Proof.* The first part of the proof is just a direct application of the Cauchy-Schwarz inequality,

$$\begin{aligned} L^2(\gamma) &= \left( \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}\, dt \right)^2 = \left( \int_a^b 1 \cdot \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}\, dt \right)^2 \\ &\leq \int_a^b dt \int_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle\, dt = (b-a)E(\gamma)\,. \end{aligned}$$

For the second part, note that the equality holds if and only if $V(t) = \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}$ and 1 are linearly dependent, which means that the speed $V(t)$ of $\gamma$ must be constant. $\square$

## 5.2 First Variation Formula

**Definition 5.4** *Let $\gamma : J \subseteq \mathbb{R} \longrightarrow M$ be a smooth immersed curve in a Riemannian manifold $M$. A variation of $\gamma$ is a map $\Gamma : (-\epsilon, \epsilon) \times J \longrightarrow M$ such that $\Gamma(0,t) = \gamma(t)$ for every $t \in J$. The vector field $W$ along $\gamma$ defined as the partial derivative $W(t) = \partial\Gamma/\partial w(0,t)$ is called the variational vector field.*

**Remark 5.5** *A variation of a curve can be understood as a family of curves $\{\Gamma(w,t)\}_{w\in(-\epsilon,\epsilon)}$ defined on the interval $J$. Similarly, $\Gamma(w,t)$ may be interpreted as a surface in $M$ with parameters $(w,t) \in (-\epsilon,\epsilon) \times J$.*

**Definition 5.6** *A variation $\Gamma$ of $\gamma : J = (a,b) \subseteq \mathbb{R} \longrightarrow M$ is called a proper variation if it fixes the end points of $\gamma$. That is, if $\Gamma(w,a) = \gamma(a)$ and $\Gamma(w,b) = \gamma(b)$ for all $w \in (-\epsilon,\epsilon)$.*

**Remark 5.7** *Equivalently, a proper variation of $\gamma$ is a variation whose variational vector field $W$ vanishes at the end points of $\gamma$. That is, $W(a) = W(b) = 0$. Such a variational vector field along $\gamma$ is called a proper variational vector field.*

**Theorem 5.8 (First Variation Formula)** *Let $\gamma : J = (a,b) \subseteq \mathbb{R} \longrightarrow M$ be a smooth immersed curve and $\Gamma(w,t)$ a proper variation of $\gamma$ with variational vector field $W(t)$. Consider the energy functional $E$ acting on the variation curves $\Gamma(w,t)$. Then,*

$$\delta E(\gamma) = \delta|_{w=0} E(\Gamma) = -2 \int_a^b \langle \nabla_{\dot\gamma}\dot\gamma(t), W(t)\rangle \, dt \, ,$$

*where $\nabla$ represents the Levi-Civita connection of $M$.*

*Proof.* We begin by proving the variation of the square of the speed of $\gamma$. Denote by $V^2(w,t) = \langle \dot\Gamma(w,t), \dot\Gamma(w,t)\rangle$ where $\dot\Gamma = \partial\Gamma/\partial t$. (Throughout this proof, for simplicity, we will avoid explicitly writing the dependence on $w$ and $t$). Then,

$$W(V^2) = W\left(\langle \dot\Gamma, \dot\Gamma\rangle\right) = 2\langle \nabla_W \dot\Gamma, \dot\Gamma\rangle \, ,$$

where $\nabla$ is the Levi-Civita connection of $M$. Hence, since $\nabla$ is symmetric,

$$\nabla_W \dot\Gamma - \nabla_{\dot\Gamma} W = [W, \dot\Gamma] = 0 \, .$$

Using this in $W(V^2)$, we have

$$W(V^2) = 2\langle \nabla_{\dot\Gamma} W, \dot\Gamma\rangle \, .$$

We next compute the first variation formula of the energy functional,

$$\begin{aligned} \delta E(\Gamma) &= \frac{\partial}{\partial w} E(\Gamma) = \frac{\partial}{\partial w}\int_a^b \langle \dot\Gamma, \dot\Gamma\rangle \, dt = \int_a^b W\left(\langle \dot\Gamma, \dot\Gamma\rangle\right) dt = \int_a^b W(V^2)\, dt \\ &= 2\int_a^b \langle \nabla_{\dot\Gamma} W, \dot\Gamma\rangle \, dt = -2\int_a^b \langle \nabla_{\dot\Gamma}\dot\Gamma, W\rangle \, dt \, , \end{aligned}$$

where in the last equality we have integrated by parts and use that $\Gamma$ is a proper variation. Finally, evaluating above expression at $w = 0$, we conclude that

$$\delta E(\gamma) = \delta|_{w=0} E(\Gamma) = -2 \int_a^b \langle \nabla_{\dot\gamma} \dot\gamma, W \rangle \, dt \,,$$

as claimed. □

**Definition 5.9** *A smooth immersed curve $\gamma$ is a critical point of the energy functional $E$ if for every proper variation of $\gamma$, $\delta E(\gamma) = 0$ holds.*

**Corollary 5.10** *A smooth immersed curve $\gamma : J \subseteq \mathbb{R} \longrightarrow M$ is a geodesic if and only if $\gamma$ is a critical point of the energy functional $E$.*

*Proof.* Let $\gamma$ be a geodesic. By definition, $\nabla_{\dot\gamma}\dot\gamma(t) = 0$ holds for every $t \in J$. Consequently, for every proper variation of $\gamma$, the first variation formula $\delta E(\gamma)$ (see Theorem 5.8) vanishes. Hence, according to Definition 5.9, $\gamma$ is a critical point of the energy functional $E$.

Conversely, let $\gamma$ be a critical point of the functional $E$. This means that $\delta E(\gamma) = 0$ for every proper variation of $\gamma$. In other words, for every proper variational vector field $W(t)$. Consider proper variations of the type $W(t) = \varphi(t)\nabla_{\dot\gamma}\dot\gamma(t)$, where $\varphi \in \mathcal{C}_o^\infty(J)$ is any smooth function which has compact support in $J$. We then have,

$$0 = -2 \int_a^b \langle \nabla_{\dot\gamma}\dot\gamma(t), \nabla_{\dot\gamma}\dot\gamma(t) \rangle \, \varphi(t) \, dt \,,$$

for every compactly supported smooth function $\varphi \in \mathcal{C}_o^\infty(J)$. It then follows from the Fundamental Lemma of Calculus of Variations that

$$\langle \nabla_{\dot\gamma}\dot\gamma(t), \nabla_{\dot\gamma}\dot\gamma(t) \rangle = 0 \,,$$

for all $t \in J$. Therefore, $\nabla_{\dot\gamma}\dot\gamma(t) = 0$ for all $t \in J$ and $\gamma$ is a geodesic. □

**Remark 5.11** *In general, the Fundamental Lemma of Calculus of Variations gives us the equations characterizing the critical points*[10] *of the corresponding functional. These equations are known as the Euler-Lagrange equations. In our particular case, the Euler-Lagrange equation associated with the energy functional $E$ is nothing but the equation of geodesics, namely, $\nabla_{\dot\gamma}\dot\gamma(t) = 0$ for all $t \in J$.*

**Definition 5.12** *A smooth immersed curve $\gamma : J \subseteq \mathbb{R} \longrightarrow M$ in a Riemannian manifold is a (length) minimizing curve if $L(\gamma) \leq L(\widetilde{\gamma})$ for any other curve $\widetilde{\gamma}$ with the same endpoints. The curve $\gamma$ is a locally minimizing curve if for every $t_o \in J$, there exists a neighborhood $U = (t_o - \epsilon, t_o + \epsilon) \subseteq J$ such that the restriction of $\gamma$ to $U$, $\gamma|_U$, is a minimizing curve among every pair of points.*

10. Critical points are also known as stationary points.

**Remark 5.13** *It follows directly from the definitions that every minimizing curve is locally minimizing.*

**Theorem 5.14** *Every geodesic in a Riemannian manifold is locally minimizing.*

**Remark 5.15** *Geodesics may not be (globally) minimizing curves. For a counterexample, see Exercise 4 of Section 5.4.*

**Proposition 5.16** *Let $p, q \in M$ be two arbitrary points of a Riemannian manifold $M$ and consider a minimizing geodesic $\gamma : J = (a, b) \subseteq \mathbb{R} \longrightarrow M$ joining $p$ to $q$, that is, $\gamma(a) = p$ and $\gamma(b) = q$. Then, for any other curve $\widetilde{\gamma}$ joining $p$ to $q$,*

$$E(\gamma) \leq E(\widetilde{\gamma}),$$

*where $E$ is the energy functional. Moreover, equality holds if and only if $\widetilde{\gamma}$ is also a minimizing geodesic.*

*Proof.* From the Cauchy-Schwarz inequality, we obtained in Proposition 5.3 that $L^2(\gamma) \leq (b-a)E(\gamma)$ for every smooth immersed curve $\gamma : J = (a, b) \subseteq \mathbb{R} \longrightarrow M$. Moreover, in the case that $\gamma$ has constant speed we had the equality. Combining this with $\gamma$ being a minimizing geodesic, we obtain

$$(b-a)E(\gamma) = L^2(\gamma) \leq L^2(\widetilde{\gamma}) \leq (b-a)E(\widetilde{\gamma}).$$

Observe that the first relation is an equality because geodesics have constant speed (see Proposition 4.21), while $L^2(\gamma) \leq L^2(\widetilde{\gamma})$ follows from the definition of a minimizing curve. This proves the first part of the statement.

For the second part, assume that $E(\gamma) = E(\widetilde{\gamma})$. This means that all the inequalities above are, indeed, equalities. From $L^2(\widetilde{\gamma}) = (b-a)E(\widetilde{\gamma})$ and Proposition 5.3, we deduce that $\widetilde{\gamma}$ is parameterized with constant speed. Furthermore, $L^2(\gamma) = L^2(\widetilde{\gamma})$, and so $\widetilde{\gamma}$ is a length minimizing curve. Therefore, from the first variation formula we conclude that it must be a geodesic. $\square$

## 5.3 Second Variation Formula

**Theorem 5.17 (Second Variation Formula)** *Let $\gamma : J = (a, b) \subseteq \mathbb{R} \longrightarrow M$ be a geodesic and $\Gamma(w, t)$ a proper variation of $\gamma$ with variational vector field $W(t)$. Consider the energy functional $E$ acting on the variation curves $\Gamma(w, t)$. Then,*

$$\delta^2 E(\gamma) = \delta|_{w=0}(\delta E)(\Gamma) = -2\int_a^b \langle \nabla^2_{\dot{\gamma}} W(t) + R(\dot{\gamma}(t), W(t))\dot{\gamma}(t), W(t)\rangle \, dt,$$

*where $\nabla$ is the Levi-Civita connection and $R$ is the Riemann curvature tensor of $M$.*

*Proof.* Following with the computations of the first variation formula (see Theorem 5.8) and differentiating again, we obtain

$$\begin{aligned}\delta^2 E(\Gamma) &= \delta(\delta E)(\Gamma) = -2\frac{\partial}{\partial w}\int_a^b \langle \nabla_{\dot\Gamma}\dot\Gamma, W\rangle\, dt = -2\int_a^b W\left(\langle \nabla_{\dot\Gamma}\dot\Gamma, W\rangle\right) dt \\ &= -2\int_a^b \left(\langle \nabla_W\nabla_{\dot\Gamma}\dot\Gamma, W\rangle + \langle \nabla_{\dot\Gamma}\dot\Gamma, \nabla_W W\rangle\right) dt\,.\end{aligned}$$

At this point, we notice that the second term in the last integral will not play any role since it will vanish at $w=0$. Indeed, at $w=0$ we will get $\langle \nabla_{\dot\gamma}\dot\gamma, \nabla_W W\rangle = 0$ because $\gamma$ is a geodesic. On the other hand, for the first term we will employ the definition of the Riemann curvature tensor $R$ (Definition 2.24), from which we deduce that

$$\begin{aligned}\nabla_W\nabla_{\dot\Gamma}\dot\Gamma &= \nabla_{\dot\Gamma}\nabla_W\dot\Gamma - \nabla_{[\dot\Gamma,W]}\dot\Gamma + R(\dot\Gamma, W)\dot\Gamma \\ &= \nabla_{\dot\Gamma}\nabla_W\dot\Gamma + R(\dot\Gamma, W)\dot\Gamma\,,\end{aligned}$$

where we have used that the second term is zero since $[\dot\Gamma, W] = 0$. Moreover, in the first term above we use that $\nabla$ is symmetric (as in the first variation formula), to get

$$\nabla_{\dot\Gamma}\nabla_W\dot\Gamma = \nabla_{\dot\Gamma}\nabla_{\dot\Gamma}W = \nabla^2_{\dot\Gamma}W\,.$$

Combining everything and evaluating $\delta^2E(\Gamma)$ at $w=0$, we obtain

$$\delta^2E(\gamma) = \delta^2|_{w=0}E(\Gamma) = -2\int_a^b \langle \nabla^2_{\dot\gamma}W + R(\dot\gamma, W)\dot\gamma, W\rangle\, dt\,,$$

proving the result. □

**Definition 5.18** *A vector field $\mathcal{J}$ along a geodesic $\gamma$ that satisfies the Jacobi equation*

$$\nabla^2_{\dot\gamma}\mathcal{J} + R(\dot\gamma, \mathcal{J})\dot\gamma = 0\,,$$

*is called a <u>Jacobi field</u>.*

**Theorem 5.19** *Let $\gamma$ be a geodesic and $\mathcal{J}$ a vector field along $\gamma$. If $\mathcal{J}$ is the variational vector field of a variation $\Gamma(w,t)$ through geodesics (that is, $\Gamma(w,t)$ is a geodesic for every $w \in (-\epsilon, \epsilon)$), then $\mathcal{J}$ is a Jacobi field.*

*Proof.* Let $\mathcal{J}$ be the variational vector field of $\Gamma(w,t)$ through geodesics. From the definition of the Riemann curvature tensor (Definition 2.24), we have

$$R(\dot\Gamma, \mathcal{J})\dot\Gamma = \nabla_{\mathcal{J}}\nabla_{\dot\Gamma}\dot\Gamma - \nabla_{\dot\Gamma}\nabla_{\mathcal{J}}\dot\Gamma + \nabla_{[\dot\Gamma,\mathcal{J}]}\dot\Gamma\,,$$

where, as usual, $\dot\Gamma = \partial\Gamma/\partial t$. The first term is zero since all the curves in the variation are geodesics and so $\nabla_{\dot\Gamma}\dot\Gamma = 0$. On the other hand, $[\dot\Gamma, \mathcal{J}] = 0$ and, hence, the last term is also zero. It remains to work just with the second term.

We will use that the Levi-Civita connection $\nabla$ is symmetric to obtain,

$$R(\dot{\Gamma},\mathcal{J})\dot{\Gamma} = -\nabla_{\dot{\Gamma}}\nabla_{\mathcal{J}}\dot{\Gamma} = -\nabla_{\dot{\Gamma}}\left(\nabla_{\dot{\Gamma}}\mathcal{J} + [\mathcal{J},\dot{\Gamma}]\right) = -\nabla^2_{\dot{\Gamma}}\mathcal{J}\,,$$

where we have used once again that $\mathcal{J}$ and $\dot{\Gamma}$ commute, that is, $[\dot{\Gamma},\mathcal{J}] = 0$.

Finally, we evaluate at $w = 0$, to conclude that

$$R(\dot{\gamma},\mathcal{J})\dot{\gamma} = -\nabla^2_{\dot{\gamma}}\mathcal{J}\,,$$

and, therefore, the Jacobi equation is satisfied. $\square$

**Remark 5.20** *Every Jacobi field can be understood as the variational vector field of a variation through geodesics.*

**Theorem 5.21 (Existence and Uniqueness of Jacobi Fields)** *Let $\gamma : J = (a,b) \subseteq \mathbb{R} \longrightarrow M$ be a geodesic and $p = \gamma(a)$. For every pair of vectors $X_p, Y_p \in T_pM$, there exists a unique Jacobi field $\mathcal{J}$ along $\gamma$ satisfying the initial conditions*

$$\mathcal{J}(a) = X_p\,, \qquad \nabla_{\dot{\gamma}}\mathcal{J}(a) = Y_p\,.$$

**Definition 5.22** *Let $\gamma$ be a geodesic joining two points $p, q \in M$. We will say that $q$ is a <u>conjugate point</u> to $p$ along $\gamma$ if there exists a Jacobi field along $\gamma$ vanishing at $p$ and $q$ but not identically zero. The dimension of the space of such Jacobi fields is the <u>multiplicity</u> (or, <u>order</u>) of the conjugacy.*

**Theorem 5.23** *Let $p \in M$ be a point in a Riemannian manifold $M$ and $X_p \in T_pM$. Then, the exponential map $\exp_p$ is a local diffeomorphism in a neighborhood of $X_p$ if and only if $q = \exp_p(X_p)$ is not a conjugate point to $p$ along the geodesic $\gamma(t) = \exp_p(tX_p)$, $t \in [0,1]$.*

**Theorem 5.24 (Bonnet-Myers' Theorem)** *Let $M$ be a complete Riemannian manifold and assume that its Ricci curvature satisfies*

$$\mathrm{Ric}_p\,(X_p) \geq \frac{1}{R^2} > 0\,,$$

*for all $p \in M$ and every unitary vector $X_p \in T_pM$. Then, $M$ is compact and its diameter (that is, the supremum of the distances among every pair of points) is smaller or equal $\pi R$.*

**Theorem 5.25 (First Jacobi Theorem)** *Let $\gamma : J = (a,b) \subseteq \mathbb{R} \longrightarrow M$ be a geodesic such that $\gamma(b)$ is not a conjugate point to $\gamma(a)$. Then, $\gamma$ is a minimizing curve, among all the variation curves for every proper variation of $\gamma$, if and only if it has no conjugate points in the interval $(a,b]$.*

**Remark 5.26** *It is essential to highlight that, under the lack of conjugate points, the geodesic* $\gamma$ *minimizes the length among all curves which can be obtained by varying* $\gamma$*, that is, is a local minimum of the length functional. It may not be a (globally) minimizing curve. For a counterexample, consider any parallel on a cylinder and recall that surfaces with non-positive Gaussian curvature do not have conjugate points (see Exercise 4 of Section 5.4).*

**Remark 5.27** *The fact that* $\gamma(b)$ *is not conjugate to* $\gamma(a)$ *makes the geodesic* $\gamma$ *an strict local minimum of the length functional. On the contrary, consider the sphere* $\mathbb{S}^2$*. The north and south poles are conjugate points and any meridian (which are geodesics as shown in Exercise 7 of Section 4.4) joining these poles have the same length. Meridians can be deformed one into another and, hence, none of them is an strict local minimum of the length functional.*

## 5.4 Exercises

1. * Let $\gamma : J = (a,b) \subseteq \mathbb{R} \longrightarrow M$ be a smooth immersed curve and $\Gamma(w,t)$ a proper variation of $\gamma$ with variational vector field $W(t)$. Consider the length functional $L$ (Definition 5.1) acting on the variation curves $\Gamma(w,t)$.

   i) Compute the first variation formula $\delta L(\gamma)$ for the length functional $L$.

   ii) Show that a curve is a critical point of the length functional ($\delta L(\gamma) = 0$ for every proper variation of $\gamma$) if and only if

   $$\nabla_{\dot\gamma}\left(\frac{\dot\gamma}{\sqrt{\langle\dot\gamma,\dot\gamma\rangle}}\right) = 0\,,$$

   holds.

   iii) Prove that a curve $\gamma$ is a geodesic if and only if it is a constant speed critical point of the length functional. (Cf. Corollary 5.10.)

2. * Let $\gamma : J = (a,b) \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ be a smooth immersed curve and $\Gamma(w,t)$ a proper variation of $\gamma$ with variational vector field $W(t)$. Consider the functional

   $$G(\gamma) = \int_a^b \langle\gamma(t),\partial_y\rangle\sqrt{\langle\dot\gamma(t),\dot\gamma(t)\rangle}\,dt\,,$$

   where $(x,y)$ are the standard coordinates of $\mathbb{R}^2$, acting on the variation curves. The functional $G$ measures the gravitational potential energy of a chain, supported at its ends, in a constant gravitational field. Note that the height is computed with respect to the $x$-axis, that is, $\langle\gamma(t),\partial_y\rangle$. The critical points of $G(\gamma)$ are called catenaries.

   i) Compute the first variation formula $\delta G(\gamma)$ for the functional $G$.

   ii) Show that a curve $\gamma$ is a critical point of $G$ (that is, a catenary) if and only if the curvature $\kappa$ of its arc length reparameterization satisfies

   $$\kappa = \frac{\langle N,\partial_y\rangle}{\langle\gamma,\partial_y\rangle}\,,$$

where $N$ is the unit normal.

iii) Let $S$ be the surface of revolution in $\mathbb{R}^3$ obtained by rotating a catenary around the $x$-axis. Show that $S$ is minimal. The surface $S$ is known as the catenoid.

3. * Let $\gamma : J = (a,b) \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ be a smooth immersed curve and $\Gamma(w,t)$ a proper variation of $\gamma$ with variational vector field $W(t)$. Consider the functional

$$T(\gamma) = \int_a^b \frac{1}{\sqrt{\langle \gamma(t), \partial_y \rangle}} \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}\, dt\,,$$

acting on the variation curves. According to the conservation of energy, the kinetic energy and the gravitational potential energy must be equal at each point of the trajectory, that is, $mV^2/2 = mg\langle \gamma(t), \partial_y \rangle$, which implies $V(t) = \sqrt{2g\langle \gamma(t), \partial_y \rangle}$. Consequently, since the speed is $ds/dt = V$, the functional $T$ measures the time needed to travel from one point to another when only constant gravity is acting. The critical points of $T$ are called brachistochrone curves (roughly speaking, the curves with shortest time).

i) Compute the first variation formula $\delta T(\gamma)$ for the functional $T$.

ii) Compute the Euler-Lagrange equation associated with $T$ and show that critical points (that is, brachistochrone curves) are cycloids.

4. Let $S$ be a circular cylinder in $\mathbb{R}^3$ parameterized by

$$x^{-1}(\theta, z) = (\cos\theta, \sin\theta, z)\,,$$

where $\theta \in (0, 2\pi)$ and $z \in \mathbb{R}$.

i) Show that the curves $z = z_o$ constant (parallels) are geodesics. (Cf. Exercise 7 of Section 4.4.)

ii) Compute the length of the curve $\gamma$ given by $z = z_o$ constant between every two points in that parallel.

iii) Prove that the curve $\gamma$ minimizes the length (is a globally minimizing curve) if and only if the length of the curve is smaller or equal $\pi$.

5. * Let $M$ be a Riemannian manifold and $f \in \mathcal{C}^\infty(M)$ such that $\langle \operatorname{grad} f, \operatorname{grad} f \rangle = 1$ at every point $p \in M$ (the gradient vector field $\operatorname{grad} f$ was defined in Exercise 2 of Section 2.5). The integral curves of $\operatorname{grad} f$ are geodesics (see Exercise 8 of Section 4.4). Show that for every pair of points $p, q \in M$, they are indeed minimizing geodesics.

6. Let $(U, x)$ be a local chart for a Riemannian manifold $M$ and consider a geodesic $\gamma : J \subseteq \mathbb{R} \longrightarrow M$ such that $\gamma(J) \subset U$.

i) Write the local equations (in terms of the coordinate functions $x_k(t)$, $k = 1, ..., n$, of $\gamma$) for the Jacobi fields along $\gamma$.

ii) Transform the second order differential equations into a system of first order equations by introducing suitable new variables.

Then, the standard theory of ordinary differential equations shows the local existence and uniqueness of Jacobi fields (Theorem 5.21).

# 6 Application to Image Reconstruction

(For more details about sub-Riemannian geometry, see [12]. For more details about the application to image reconstruction, see [1] and references therein. For more details about functionals depending on the curvature, see [16].)

## 6.1 Sub-Riemannian Manifold

**Definition 6.1** *Let $M$ be a smooth manifold. A sub-bundle of the tangent bundle $TM$ is called a distribution $\mathfrak{D}$ on $M$. A distribution $\mathfrak{D}$ is bracket-generating if for every $p \in M$, the sections of $\mathcal{D}$ near $p$ together with all their Lie brackets span the tangent space $T_pM$.*

**Definition 6.2** *Let $M$ be a smooth manifold and $\mathfrak{D}$ a distribution on $M$. A sub-Riemannian metric is a smoothly varying positive definite bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{D}$.*

**Definition 6.3** *A smooth manifold $M$ equipped with a sub-Riemannian metric $\langle\cdot,\cdot\rangle$ on a bracket-generating distribution $\mathfrak{D}$ is a sub-Riemannian manifold.*

**Remark 6.4** *If the distribution $\mathfrak{D}$ is equal to the whole tangent bundle $TM$, then $\langle\cdot,\cdot\rangle$ is a Riemannian metric and $M$ is a Riemannian manifold. Having a sub-Riemannian manifold (in contrast to a Riemannian manifold) just means, roughly speaking, that we can only compute the magnitude of some vector fields and, hence, the length of some special curves.*

**Definition 6.5** *Let $M$ be a sub-Riemannian manifold. A $\mathfrak{D}$-curve on $M$ is a smooth immersed curve $\delta : J = (a,b) \subseteq \mathbb{R} \longrightarrow M$ which is always tangent to the distribution $\mathfrak{D}$. That is, $\dot{\delta}(t) \in \mathfrak{D}_{\delta(t)}$ for all $t \in J$.*

**Remark 6.6** *The notion of a $\mathfrak{D}$-curve does not need that the distribution is bracket-generating and, indeed, we can consider $\mathfrak{D}$-curves simply on a smooth manifold $M$ with a (not necessarily bracket-generating) distribution. The Theorem of Chow-Rashevskii (see Chapter 2 of [12]) states that, if $\mathfrak{D}$ is bracket-generating, then there is a $\mathfrak{D}$-curve joining every two points of $M$. Hence, we will restrict ourselves to bracket-generating distributions.*

**Definition 6.7** *Let $M$ be a sub-Riemannian manifold and $\delta : J = (a,b) \subseteq \mathbb{R} \longrightarrow M$ a $\mathfrak{D}$-curve. The length of $\delta$ is*

$$L(\delta) = \int_a^b \sqrt{\langle\dot{\delta}(t),\dot{\delta}(t)\rangle}\,dt\,.$$

*For every $p, q \in M$, the distance between $p$ and $q$ is the infimum of the lengths of all $\mathfrak{D}$-curves joining $p$ and $q$.*

## 6.2 Sub-Riemannian Model

**Remark 6.8** *Neurobiological research over the past few decades has greatly clarified the functional mechanisms of the first layer V1 of the visual cortex (also known as the primary visual cortex). Such layer contains a variety of types of cells, including the so-called simple cells. Researchers found that V1 constitutes of orientation selective cells at all orientations for all retinal positions, so simple cells are sensitive to orientation-specific brightness gradients.*

*Recently, this structure of the primary visual cortex has been modeled using sub-Riemannian geometry. In particular, the unit tangent bundle of the plane* $\mathbb{R}^2 \times \mathbb{S}^1$ *can be used as an abstraction to study the organization and mechanisms of V1.*

*According to this model for V1, in the space* $\mathbb{R}^2 \times \mathbb{S}^1$*, each point* $(x, y, \theta)$ *represents a column of cells associated with a point of retinal data* $(x, y) \in \mathbb{R}^2$*, all of which are adjusted to the orientation given by the angle* $\theta \in \mathbb{S}^1$*. More specifically, the vector* $(\cos\theta, \sin\theta)$ *is the direction of maximal rate of change of brightness at a point* $(x, y)$ *of the picture seen by the eye. Thus, when the cortex cells are stimulated by an image, the border of the image gives a curve inside the 3D-space* $\mathbb{R}^2 \times \mathbb{S}^1$*, but such curves are restricted to be tangent to the distribution spanned by the vector fields*

$$X_1 = \cos\theta\,\partial_x + \sin\theta\,\partial_y\,, \qquad X_2 = \partial_\theta\,.$$

**Definition 6.9** *We will call the* <u>*unit tangent bundle of the plane*</u> *to the sub-Riemannian manifold* $(\mathbb{R}^2 \times \mathbb{S}^1, \mathfrak{D}, \langle\cdot,\cdot\rangle)$*, where* $\mathfrak{D}$ *is the (bracket-generating) distribution spanned by*

$$X_1 = \cos\theta\,\partial_x + \sin\theta\,\partial_y\,, \qquad X_2 = \partial_\theta\,,$$

*and where* $\langle\cdot,\cdot\rangle$ *is the metric on* $\mathfrak{D}$ *defined by requesting that* $X_1$ *and* $X_2$ *are everywhere orthonormal.*

**Remark 6.10** *It is believed that if a piece of the contour of a picture is missing to the eye vision (or maybe it is covered by an object), then the brain tends to "complete" the curve by minimizing the length between the endpoints. In other words, the brain considers a sub-Riemannian minimizing geodesic between the endpoints of the missing data.*

**Theorem 6.11** *Let* $(\mathbb{R}^2 \times \mathbb{S}^1, \mathfrak{D}, \langle\cdot,\cdot\rangle)$ *be the unit tangent bundle of the plane and consider two points* $p_o = (x_o, y_o, \theta_o)$ *and* $p_1 = (x_1, y_1, \theta_1)$ *in* $\mathbb{R}^2 \times \mathbb{S}^1$*. The* $\mathfrak{D}$*-curves* $\delta : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2 \times \mathbb{S}^1$ *satisfying* $\dot{y}(t) = \dot{x}(t)\cos\theta(t)$ *that cover the distance between* $p_o$ *and* $p_1$ *(that is, the sub-Riemannian minimizing geodesics) are the lifts of curves* $\gamma : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ *in the plane that minimize the functional*

$$\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa^2(s) + 1}\, ds\,,$$

*among all the curves in* $\mathbb{R}^2$ *joining* $(x_o, y_o)$ *to* $(x_1, y_1)$ *and with initial and final angles between* $\dot{\gamma}$ *and the x-axis,* $\theta_o$ *and* $\theta_1$*, respectively. (Recall that s denotes the arc length parameter of* $\gamma$ *and that* $\kappa$ *is the curvature of* $\gamma$*.)*

*Proof.* Every $\mathfrak{D}$-curve given by $\delta(t) = (x(t), y(t), \theta(t))$ in the unit tangent bundle of the plane $\mathbb{R}^2 \times \mathbb{S}^1$ with $\dot{y}(t) = \dot{x}(t)\cos\theta(t)$ is the lift of a smooth curve $\gamma(t) = (x(t), y(t))$ in the plane $\mathbb{R}^2$ whose tangent vector $\dot{\gamma}(t)$ makes the angle $\theta(t)$ with the $x$-axis, that is,

$$\dot{\gamma}(t) = V(t)\cos\theta\,\partial_x + V(t)\sin\theta\,\partial_y = V(t)X_1\,,$$

where $V(t) = \sqrt{\langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle}$ is the speed of $\gamma(t)$.

Conversely, every curve $\gamma(t) = (x(t), y(t))$ in the plane $\mathbb{R}^2$ can be lifted to a $\mathfrak{D}$-curve $\delta(t) = (x(t), y(t), \theta(t))$ in $\mathbb{R}^2 \times \mathbb{S}^1$ by setting $\theta(t)$ equal to the angle between $\dot{\gamma}(t)$ and the $x$-axis.

Now, the tangent vector $\dot{\delta}(t)$ of the $\mathfrak{D}$-curve $\delta(t)$ satisfies

$$\langle\dot{\delta}(t), \dot{\delta}(t)\rangle = \langle V(t)X_1 + \dot{\theta}(t)X_2, V(t)X_1 + \dot{\theta}(t)X_2\rangle = V^2(t) + \dot{\theta}^2(t)\,,$$

where $X_2 = \partial_\theta$ and we have used the definition of the sub-Riemannian metric $\langle\cdot,\cdot\rangle$ of the unit tangent bundle of the plane. (Remember that the vector fields $X_1$ and $X_2$ are everywhere orthonormal.) We also recall now that for a fixed direction in $\mathbb{R}^2$ and a curve $\gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ the derivative with respect to the arc length parameter $s \in I$ of the angle between $\gamma'(s)$ and the fixed direction is the curvature of $\gamma$, perhaps up to a sign (cf. Exercise 3 of Section 4.4). In our case, $(\theta'(s))^2 = \kappa^2(s)$ since $\theta$ is the angle between $\dot{\gamma}$ and the $x$-axis. Therefore, applying the chain rule, we have

$$\langle\dot{\delta}(t), \dot{\delta}(t)\rangle = V^2(t) + V^2(t)\left(\frac{\dot{\theta}(t)}{V(t)}\right)^2 = V^2(t)\left(1 + \kappa^2(t)\right),$$

where $\kappa(t)$ is the curvature of $\gamma$ in the parameter $t \in J = (a, b)$. It then follows that

$$L(\delta) = \int_a^b \sqrt{\langle\dot{\delta}(t), \dot{\delta}(t)\rangle}\,dt = \int_a^b \sqrt{\kappa^2(t) + 1}\,V(t)\,dt = \mathbf{\Theta}(\gamma)\,,$$

after a change of variable.

Consequently, the $\mathfrak{D}$-curves $\delta$ with $\dot{y}(t) = \dot{x}(t)\cos\theta(t)$ that cover the distance between two points $p_o$ and $p_1$ in $\mathbb{R}^2 \times \mathbb{S}^1$ are the lifts of curves $\gamma$ in the plane joining $(x_o, y_o)$ to $(x_1, y_1)$ with initial angle $\theta_o$ and final angle $\theta_1$ that minimize the functional $\mathbf{\Theta}$. $\square$

**Remark 6.12** *The hypercolumnar organization of the primary visual cortex suggests that the cost of moving one orientation unit is not necessarily the same as to moving spatial units, then the image reconstruction problem should consider minimizing the more general functional*

$$\mathbf{\Theta}_\mu(\gamma) = \int_\gamma \sqrt{\kappa^2(s) + \mu^2}\,ds\,,$$

*where $\mu \in \mathbb{R}$ is a non-zero real constant.*

## 6.3 Total Curvature Type Energies

**Theorem 6.13** *Let $\gamma : I = (0, L) \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ be an arc length parameterized smooth immersed curve and consider a proper variation of $\gamma$ with variational vector field $W$. Consider the functional $\mathbf{\Theta}_\mu$ acting on the variation curves. Then,*

$$\delta\mathbf{\Theta}_\mu(\gamma) = \int_\gamma \langle \mathcal{E}(s), W(s)\rangle \, ds + \mathcal{B}\left(\gamma, W(s)\right|_0^L \,,$$

*where*

$$\mathcal{E}(s) = \nabla^2_{\gamma'}\left(\frac{1}{\sqrt{\kappa^2(s)+\mu^2}}\nabla_{\gamma'}\gamma'(s)\right) + \nabla_{\gamma'}\left(\frac{\kappa^2(s)-\mu^2}{\sqrt{\kappa^2(s)+\mu^2}}\gamma'(s)\right),$$

*is the Euler-Lagrange vector field and*

$$\mathcal{B}(\gamma, W(s)) = \frac{1}{\sqrt{\kappa^2(s)+\mu^2}}\langle \nabla_{\gamma'}\gamma'(s), \nabla_{\gamma'}W(s)\rangle \,,$$

*is the boundary vector field.*

*Proof.* Consider a proper variation $\Gamma$ of $\gamma$. In order to compute the first variation formula for $\mathbf{\Theta}_\mu$, we need to reparameterize all the curves in the variation by an arbitrary parameter $t$, so that all of them are defined on the same interval, say $t \in J = (a, b)$. In other words, $\Gamma : (-\epsilon, \epsilon) \times J \longrightarrow \mathbb{R}^2$ and $t \in J$ is arbitrary. Then,

$$\mathbf{\Theta}_\mu(\Gamma(w,t)) = \int_a^b \sqrt{\kappa^2(w,t)+\mu^2}\, V(w,t)\, dt \,,$$

where $V(w,t) = \sqrt{\langle \dot\Gamma(w,t), \dot\Gamma(w,t)\rangle}$ is the speed of $\Gamma(w,t)$ and $\dot\Gamma = \partial\Gamma/\partial t$. (As customary in long computations, we will avoid writing the dependence on $w$ and $t$.)

In Theorem 5.8 we computed the variation of the square of the speed of $\gamma$, obtaining

$$W(V^2) = 2\langle \nabla_{\dot\Gamma} W, \dot\Gamma\rangle \,.$$

Hence, since $W(V^2) = 2VW(V)$, we get

$$W(V) = \frac{\langle \nabla_{\dot\Gamma} W, \dot\Gamma\rangle}{V} \,.$$

We next need to understand the variation of $\kappa^2(w,t)$. From the definition of the curvature of a curve (Definition 4.13),

$$W(\kappa^2) = W\left\langle \nabla_T T, \nabla_T T\rangle\right) = 2\langle \nabla_W \nabla_T T, \nabla_T T\rangle \,,$$

where $T = \dot\Gamma/V$ is the unit tangent vector field along the variation curves $\Gamma$. We point out here that the variational vector field $W$ and $T$ may not commute. In fact,

$$[T, W] = [\dot\Gamma/V, W] = \frac{1}{V}[\dot\Gamma, W] - W\left(\frac{1}{V}\right)\dot\Gamma = \frac{W(V)}{V^2}\dot\Gamma = \frac{W(V)}{V}T \,,$$

where we have used that $[\dot{\Gamma}, W] = 0$. We will use this relation and the definition of the Riemann curvature tensor $R$ (Definition 2.24) to rewrite $\nabla_W \nabla_T T$. Recall that all our variation curves are in the plane $\mathbb{R}^2$, which has curvature identically zero, that is, $R(X,Y)Z = 0$ for all $X, Y, Z \in \mathfrak{X}(\mathbb{R}^2)$. Consequently,

$$\begin{aligned}\nabla_W \nabla_T T &= \nabla_T \nabla_W T - \nabla_{[T,W]} T = \nabla_T \nabla_W T - \frac{W(V)}{V} \nabla_T T \\ &= \nabla_T \left( \nabla_T W + [W,T] \right) - \frac{W(V)}{V} \nabla_T T \\ &= \nabla_T^2 W - \nabla_T \left( \frac{W(V)}{V} T \right) - \frac{W(V)}{V} \nabla_T T \,,\end{aligned}$$

where in the second line we have used the symmetry of the Levi-Civita connection $\nabla$. Therefore,

$$\begin{aligned}W(\kappa^2) &= 2\langle \nabla_W \nabla_T T, \nabla_T T\rangle = 2\langle \nabla_T^2 W, \nabla_T T\rangle - 4\frac{W(V)}{V}\langle \nabla_T T, \nabla_T T\rangle \\ &= 2\langle \nabla_T^2 W, \nabla_T T\rangle - 4\kappa^2 \langle \nabla_T W, T\rangle \,,\end{aligned}$$

since $W(V)/V = \langle \nabla_T W, T\rangle$.

We then differentiate on $\mathbf{\Theta}_\mu$ to compute its associated first variation formula

$$\begin{aligned}\delta \mathbf{\Theta}_\mu(\Gamma) &= \frac{\partial}{\partial w} \mathbf{\Theta}_\mu(\Gamma) = \frac{\partial}{\partial w} \int_a^b \sqrt{\kappa^2 + \mu^2}\, V\, dt = \int_a^b \left( \frac{W(\kappa^2)}{2\sqrt{\kappa^2+\mu^2}} V + \sqrt{\kappa^2+\mu^2}\, W(V) \right) dt \\ &= \int_a^b \left( \langle \nabla_T^2 W, \frac{1}{\sqrt{\kappa^2+\mu^2}} \nabla_T T\rangle - \langle \nabla_T W, \frac{2\kappa^2 T}{\sqrt{\kappa^2+\mu^2}}\rangle + \langle \nabla_T W, \sqrt{\kappa^2+\mu^2}\, T\rangle \right) V\, dt.\end{aligned}$$

Evaluating at $w = 0$ and reparameterizing in terms of the arc length parameter $s \in I = (0, L)$ of $\gamma$, we have

$$\begin{aligned}\delta \mathbf{\Theta}_\mu(\gamma) &= \delta|_{w=0} \mathbf{\Theta}_\mu(\Gamma) = \int_0^L \left( \langle \nabla_T^2 W, \frac{1}{\sqrt{\kappa^2+\mu^2}} \nabla_T T\rangle - \langle \nabla_T W, \frac{\kappa^2 - \mu^2}{\sqrt{\kappa^2+\mu^2}} T\rangle \right) ds \\ &= \int_0^L \left( -\langle \nabla_T W, \nabla_T \left[ \frac{1}{\sqrt{\kappa^2+\mu^2}} \nabla_T T \right]\rangle + \langle W, \nabla_T \left[ \frac{\kappa^2-\mu^2}{\sqrt{\kappa^2+\mu^2}} T \right]\rangle \right) ds \\ &\quad + \left( \langle \nabla_T W, \frac{1}{\sqrt{\kappa^2+\mu^2}} \nabla_T T\rangle \right|_0^L \\ &= \int_0^L \left( \langle \nabla_T^2 \left[ \frac{1}{\sqrt{\kappa^2+\mu^2}} \nabla_T T \right], W\rangle + \langle \nabla_T \left[ \frac{\kappa^2-\mu^2}{\sqrt{\kappa^2+\mu^2}} T \right], W, \rangle \right) ds \\ &\quad + \left( \langle \nabla_T W, \frac{1}{\sqrt{\kappa^2+\mu^2}} \nabla_T T\rangle \right|_0^L \,,\end{aligned}$$

where in the last two equalities we have integrated by parts and use that $W$ is proper, that is, $W(0) = W(L) = 0$. This concludes the proof. □

**Definition 6.14** *A smooth immersed curve $\gamma$ is a* critical point *of $\mathbf{\Theta}_\mu$ if $\delta\mathbf{\Theta}_\mu(\gamma) = 0$ holds for every proper variation of $\gamma$.*

**Corollary 6.15** *Let $\gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ be a smooth immersed curve in the plane $\mathbb{R}^2$ parameterized by the arc length $s \in I$. If $\gamma$ is a critical point of $\mathbf{\Theta}_\mu$, then*

$$\frac{d^2}{ds^2}\left(\frac{\kappa(s)}{\sqrt{\kappa^2(s)+\mu^2}}\right) - \mu^2\,\frac{\kappa(s)}{\sqrt{\kappa^2(s)+\mu^2}} = 0\,,$$

*holds along $\gamma$.*

*Proof.* Let $\gamma$ be a critical point of $\mathbf{\Theta}_\mu$. Then, $\delta\mathbf{\Theta}_\mu(\gamma) = 0$ holds for every proper variation of $\gamma$. In particular, we consider variations whose variational vector field is of the type $W(s) = \varphi(s)T(s) + \psi(s)N(s)$, where $T$ is the unit tangent to $\gamma$ and $N$ is the unit normal, and $\varphi, \psi \in \mathcal{C}^\infty_o(I)$ are compactly supported smooth functions. It then follows from the Fundamental Lemma of the Calculus of Variations, considering first $\varphi = 0$ identically and then $\psi = 0$ identically, that both the tangential and normal components of the Euler-Lagrange vector field $\mathcal{E}(s)$ must be zero. Hence, the Euler-Lagrange (vector) equation is, precisely, $\mathcal{E}(s) = 0$.

Now, using the expression of the vector field $\mathcal{E}$ and the Frenet-Serret equations (see Remark 4.18) we obtain that the tangential component of $\mathcal{E}$ is identically zero, while the vanishing of the normal component gives rise to the desired ordinary differential equation. □

**Remark 6.16** *The above proposition gives us the Euler-Lagrange equation associated with the functional $\mathbf{\Theta}_\mu$ which is a necessary, but not sufficient, condition to obtain critical points. This Euler-Lagrange equation is a second order ordinary differential equation in the curvature $\kappa$, and it can be explicitly solved obtaining*

$$\kappa(s) = \frac{\mu\left(c_1 e^{\mu s} + c_2 e^{-\mu s}\right)}{\sqrt{1-\left(c_1 e^{\mu s} + c_2 e^{-\mu s}\right)^2}}\,,$$

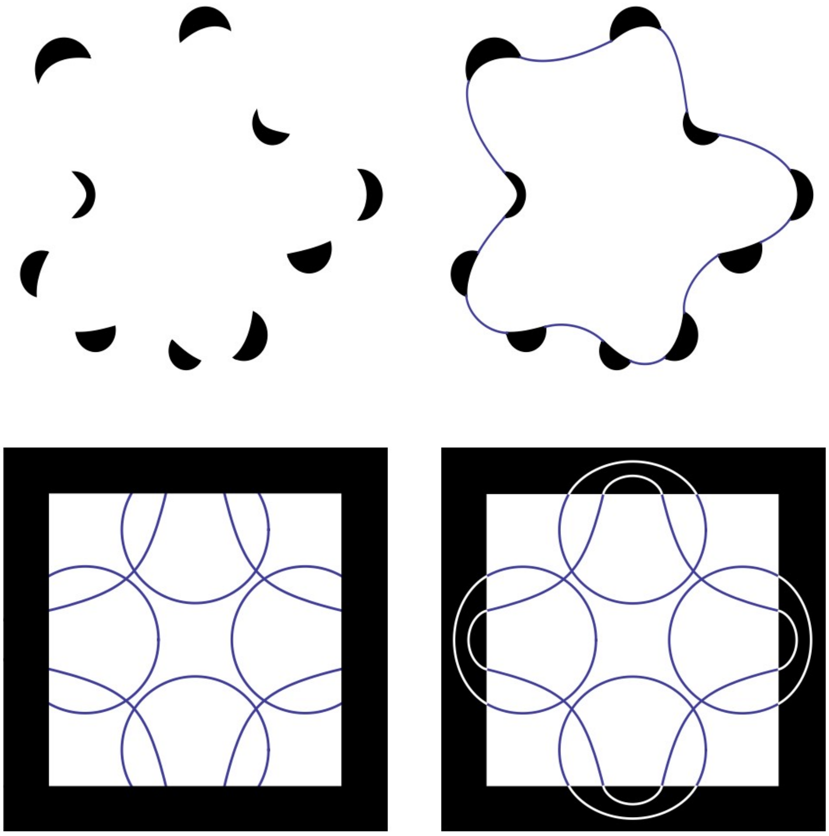
*where $c_1, c_2 \in \mathbb{R}$ are constants of integration.*

*Once the curvature is explicitly obtained it is possible to give a parameterization of the planar curves in terms of two quadratures (parameterizations by just one quadrature are also achievable, see Exercise 1 of Section 6.4). In fact, a planar curve with curvature $\kappa(s)$ can be parameterized in terms of the arc length parameter $s \in I$ as (cf. Exercise 3 of Section 4.4)*

$$\gamma(s) = \left(\int \cos\left[\int \kappa\, ds\right] ds, \int \sin\left[\int \kappa\, ds\right] ds\right).$$

*However, a specific determination of critical points of $\mathbf{\Theta}_\mu$ implies that the integration constants must be determined. This can be done by imposing the planar curve $\gamma$, parameterized as above, to satisfy the given boundary conditions, but this requires solving a highly nonlinear system.*

**Remark 6.17** *In the following figures (obtained from [16]), we show examples of image reconstruction using the above described method of minimum length in the unit tangent bundle of the plane.*



## 6.4 Exercises

1. Let $\gamma : I = (0, L) \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ be an arc length parameterized smooth immersed curve and assume that $\gamma$ is a critical point of the functional ([16])

$$\mathbf{\Theta}_\mu(\gamma) = \int_\gamma \sqrt{\kappa^2 + \mu^2}\, ds \,.$$

   i) Show that the following is a parameterization of $\gamma$ in terms of the arc length parameter $s \in I$,

$$\gamma(s) = \frac{1}{\sqrt{a}} \left( \frac{\kappa(s)}{\sqrt{\kappa^2(s) + \mu^2}}, -\int \frac{\mu^2}{\sqrt{\kappa^2(s) + \mu^2}}\, ds \right),$$

   where $a > 0$ is a constant.

   ii) Consider the surface of revolution $S$ in $\mathbb{R}^3$ obtained by rotating $\gamma(s)$ (parameterized as above) around the $x$-axis. Show that the Gaussian curvature of $S$ is

$$K = -\mu^2 \,.$$

iii) Prove that every surface of revolution $S$ in $\mathbb{R}^3$ with constant negative Gaussian curvature $K$ is obtained by rotating a critical point of $\mathbf{\Theta}_\mu$, with $\mu = \sqrt{-K}$, around a suitable axis.

2. Let $\gamma : J = (a, b) \subseteq \mathbb{R} \longrightarrow M$ be a non-geodesic smooth curve immersed in a Riemannian manifold $M$. Consider a proper variation $\Gamma(w, t)$ of $\gamma$ with variational vector field $W(t)$.

   i) Show that the variation of the speed $V(w,t) = \sqrt{\langle \dot{\Gamma}(w,t), \dot{\Gamma}(w,t)\rangle}$ in $\gamma$ in the direction of $W$ is
   $$W(V) = \langle \nabla_T W, T\rangle V\,,$$
   where $T$ is the unit tangent along $\gamma$ and $\nabla$ is the Levi-Civita connection of $M$ ([16]).

   ii) Show that the variation of the curvature $\kappa(w,t) = \sqrt{\langle \nabla_T T(w,t), \nabla_T T(w,t)\rangle}$ in $\gamma$ in the direction of $W$ is
   $$W(\kappa) = \frac{1}{\kappa}\langle \nabla_T^2 W, \nabla_T T\rangle - 2\kappa \langle \nabla_T W, T\rangle + \frac{1}{\kappa}\langle R(T, W)T, \nabla_T T\rangle\,,$$
   where $R$ is the Riemann curvature tensor of $M$ ([16]).

3. Let $\gamma : I = (0, L) \subseteq \mathbb{R} \longrightarrow M$ be an arc length parameterized curve immersed in a Riemannian manifold of dimension two with constant sectional curvature $K$. Consider a proper variation of $\gamma$ with variational vector field $W$ and assume that the functional
   $$\mathbf{\Theta}(\gamma) = \int_\gamma \left(\kappa^2 + \mu\right) ds\,,$$
   where $\mu \in \mathbb{R}$ and $\kappa$ is the curvature of $\gamma$, is acting on the variation curves. The functional $\mathbf{\Theta}$ is called the <u>bending energy</u>, [16].

   i) Compute the first variation formula $\delta\mathbf{\Theta}(\gamma)$ for the bending energy.

   ii) Show that if a curve is a critical point of the bending energy ($\delta\mathbf{\Theta}(\gamma) = 0$ for every proper variation of $\gamma$), then the Euler-Lagrange equation
   $$\kappa'' + \frac{1}{2}\kappa\left(\kappa^2 - \mu\right) + K\kappa = 0\,,$$
   holds along $\gamma$. Critical points of the bending energy are <u>elastic curves</u>.

   iii) Assume that an elastic curve $\gamma$ has constant curvature. Prove that then, either $\gamma$ is a geodesic or
   $$\kappa^2 = \mu - 2K\,.$$
   In particular, let $M = \mathbb{R}^2$ be the Euclidean plane. Show that elastic curves with constant curvature are either straight lines or circles of radius $r = 1/\sqrt{\mu}$ for $\mu > 0$.

   iv) Assume now that $\gamma$ is an elastic curve with non-constant curvature. Show that the curvature of $\gamma$ satisfies the first order ordinary differential equation
   $$(\kappa')^2 + \frac{1}{4}\left(\kappa^2 - \mu\right)^2 + K\kappa^2 = a\,,$$
   for some constant of integration $a \in \mathbb{R}$.

4. * Let $\gamma : I = (0, L) \subseteq \mathbb{R} \longrightarrow M$ be an arc length parameterized non-geodesic smooth immersed curve in a Riemannian manifold $M$ of dimension two and with constant sectional curvature $K$. Consider a proper variation $\Gamma$ of $\gamma$ with variational vector field $W$, such that none of the variation curves is a geodesic, and assume that the functional

$$\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa}\, ds\,,$$

is acting on the (non-geodesic) variation curves. This functional will be referred to as the Blaschke's functional, [16].

i) Compute the first variation formula $\delta\mathbf{\Theta}(\gamma)$ for the Blaschke's functional.

ii) Show that the curvature $\kappa$ of a critical point $\gamma$ for the Blaschke's functional must satisfy the Euler-Lagrange equation

$$\frac{d^2}{ds^2}\left(\frac{1}{\sqrt{\kappa}}\right) + \frac{1}{\sqrt{\kappa}}\left(K - \kappa^2\right) = 0\,.$$

iii) Assume that $M = \mathbb{R}^2$ is the Euclidean plane. Find the explicit expression of the curvature of a critical point and show that $\gamma$ is a catenary (cf. Exercise 2 of Section 5.4).

iv) Assume that $M = \mathbb{S}^2$ is the round sphere. Prove that for every pair of relatively prime natural numbers $(n, m)$ satisfying $m < 2n < \sqrt{2}\, m$ there exists a unique closed curve $\gamma$ critical for the Blaschke's functional $\mathbf{\Theta}$. The vertical lift via the Hopf fibration of the critical curve corresponding to $m = 5$ and $n = 3$ is the surface in $\mathbb{S}^3$ whose stereographic projection is illustrated on the first page.

5. * Let $\gamma : I = (0, L) \subseteq \mathbb{R} \longrightarrow M$ be an arc length parameterized non-geodesic smooth immersed curve in a Riemannian manifold $M$ of dimension two and with constant sectional curvature $K$. Consider a proper variation $\Gamma$ of $\gamma$ with variational vector field $W$, such that none of the variation curves is a geodesic, and assume that the functional

$$\mathbf{\Theta}(\gamma) = \int_\gamma \frac{1}{\kappa}\, ds\,,$$

is acting on the (non-geodesic) variation curves.

i) Compute the first variation formula $\delta\mathbf{\Theta}(\gamma)$.

ii) Show that the curvature $\kappa$ of a critical point $\gamma$ must satisfy the Euler-Lagrange equation

$$\frac{d^2}{ds^2}\left(\frac{1}{\kappa^2}\right) + \frac{K}{\kappa} + 2 = 0\,.$$

iii) Assume $M = \mathbb{R}^2$ is the Euclidean plane. Prove that the critical points for $\mathbf{\Theta}$ are cycloids (cf. Exercise 3 of Section 5.4).

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