Minimal Surfaces Bounded by Elastic Curves

Álvaro Pámpano Llarena

Elastic Curves and Surfaces with Applications and Numerical Representations

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- Motivation. It combines two of the most interesting energies of Geometric Calculus of Variations.

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3. Finally, from tangent variations we also get
$$J' \cdot n = 0\,, \qquad \text{on } \partial\Sigma\,.$$

Minimal Immersions

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Let $X : \Sigma \to \mathbb{R}^3$ be a minimal immersion critical for $\mathcal{W}[X]$, then the boundary $\partial\Sigma$ is composed by closed and simple elastic curves, i.e. critical for

$$\mathcal{E}[C] := \oint_C \left(\kappa^2 + \lambda\right) ds\,,$$

with $\lambda := \beta/\alpha > 0$.

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Let $C : [0, L] \to \mathbb{R}^3$ be an elastic curve, i.e. critical curve for

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https://www.youtube.com/watch?v=49CeK8g1RAo

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- We compare the genus of this particular Seifert surface with the genus of the boundary knot, obtaining $p = 1$.

Disc Type Surfaces

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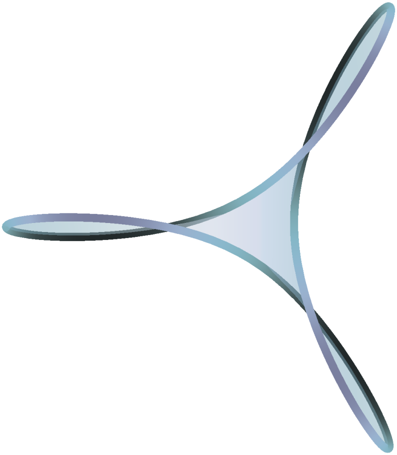

FIGURE: $C \cong G(3,1)$

Disc Type Surfaces

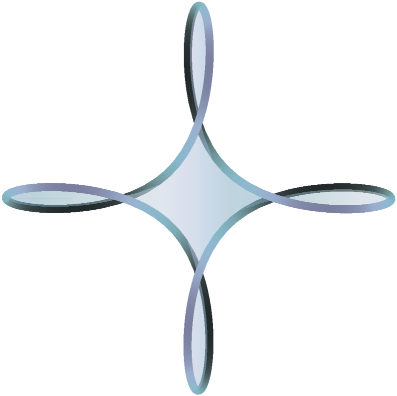

Figure: $C \cong G(4,1)$

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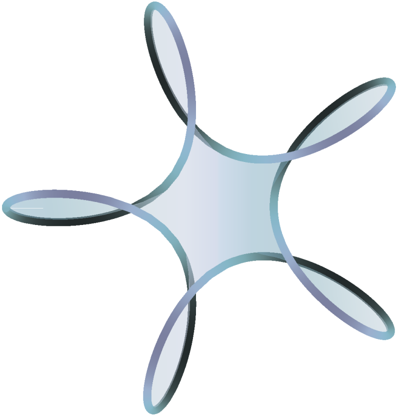
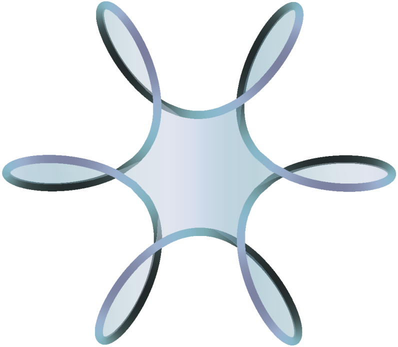


FIGURE: $C \cong G(5,1)$

Disc Type Surfaces



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- A construction involving the Plateau problem:

Annular Type Surfaces

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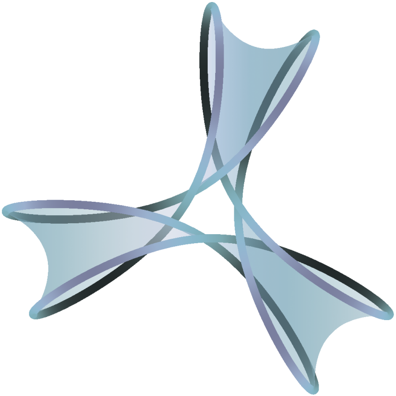

FIGURE: $C_i \cong G(3,1)$

Annular Type Surfaces

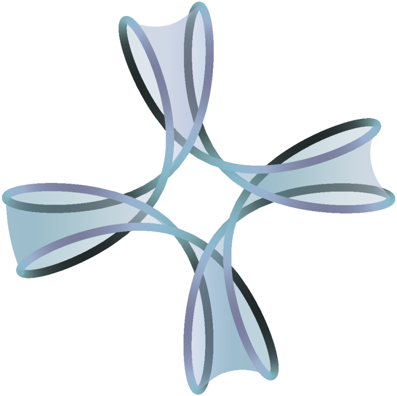

FIGURE: $C_i \cong G(4, 1)$

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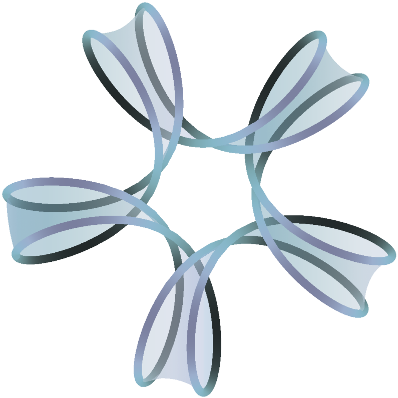

FIGURE: $C_i \cong G(5, 1)$

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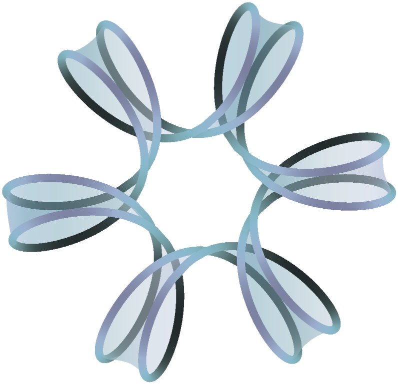

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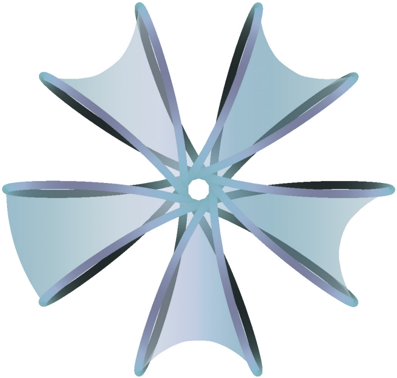

FIGURE: $C_i \cong G(5,2)$

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Recall the expression of the Euler-Willmore energy,

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3. For topological discs, the minimum is attained at a planar disc bounded by a circle of radius $\sqrt{\alpha/\beta}$. For annuli, multiple solutions (catenoid, Riemann's minimal examples,...).

THE END

- B. Palmer and A. Pámpano, Minimizing Configurations for Elastic Surface Energies with Elastic Boundaries, *submitted*.

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Thank You!