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PLANAR P-ELASTICAE AND ROTATIONAL LINEAR WEINGARTEN SURFACES

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Abstract. We variationally characterize the profile curves of rotational linear Weingarten surfaces as planar p-elastic curves. Moreover, by evolving these planar p-elasticae under the binormal flow with prescribed velocity, we describe a procedure to construct all rotational linear Weingarten surfaces, locally. Finally, we apply our findings to two well-known family of surfaces.

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1. Introduction

These notes are a printed version of the talk given by the author at the International Conference on Geometry, Integrability and Quantization held in Varna in June 2018. The purpose of the talk was to present some results included in the works [2], [3], [12] and [16]. Here, ideas and arguments are only sketched while proofs are omitted. Interested readers are going to be referred to [2], [3], [12] or [16], respectively, for a complete and more general treatment.

In classical Differential Geometry of surfaces, the intrinsic information of a surface, S, is encoded in the *first fundamental form*. On the other hand, for a surface immersed in the Euclidean 3-space, $\mathbb{R}^3$, the most important extrinsic invariant is, probably, the *mean curvature*, H, which can be computed with the aid of the *second fundamental form*. Moreover, the combination of these two fundamental forms of the immersion of S into $\mathbb{R}^3$ gives rise to the *shape operator*. The shape operator is symmetric (and, therefore, diagonalizable) and its eigenvalues are usually called *principal curvatures*.

Now, it can be checked that the mean curvature, H, verifies $2H = \kappa_1 + \kappa_2$, κ_1 and κ_2 being the principal curvatures. In [3], constant mean curvature (from now on, CMC) invariant surfaces in any Riemannian or Lorentzian 3-space form have been considered. Notice that the CMC condition expressed in terms of the principal curvatures gives rise to a relation of the type $\Upsilon(\kappa_1, \kappa_2) = 0$, for a smooth function Υ. Surfaces verifying a certain relation $\Upsilon(\kappa_1, \kappa_2) = 0$ between their principal curvatures are usually called *Weingarten surfaces*. These surfaces were introduced by Weingarten in [21] and its study occupies an important role in classical Differential Geometry.

The simplest relation of the type $\Upsilon(\kappa_1, \kappa_2) = 0$ which extends the CMC condition is the affine relation, often called *linear relation* in the literature. That is,

$$a\,\kappa_1 + b\,\kappa_2 = c\,,$$

where a, b and c are three real constants, such that, $a^2 + b^2 \neq 0$. We are going to call *linear Weingarten surfaces* to the surfaces whose principal curvatures verify this linear relation. Trivial examples appear whenever a and c (or, equivalently, b and c) are both zero, or, also when the surface has one constant principal curvature. If $\kappa_1 = 0$ (equivalently, if $\kappa_2 = 0$), then the surface is developable. Moreover, if κ_1 is a non-zero constant (equivalently, κ_2), these surfaces were classified in [20]. In particular, since we are just concerned with rotational surfaces, then they must be either spheres or torus of revolution.

After these examples, without loss of generality, we can rewrite above linear relation as

$$\kappa_1 = a\,\kappa_2 + b\,, \tag{1}$$

where $a, b \in \mathbb{R}$ and $a \neq 0$. Well-known families of linear Weingarten surfaces are the following;

- *Totally Umbilical Surfaces*. This is the case where $a = 1$ and $b = 0$.
- *Isoparametric Surfaces*. In this case both principal curvatures are constant. Besides the umbilical surfaces, the surface may be a spherical cylinder.
- *Constant Mean Curvature Surfaces*. This is the case when $a = -1$ and the surface has CMC $H = b/2$. Observe that in [3] invariant CMC surfaces have also been studied in Riemannian and Lorentzian 3-space forms by considering $H = H_o \in \mathbb{R}$ instead of above linear relation, (1).

Under this definition, a first result due to Chern proves that the sphere is the only ovaloid with the property that κ_1 is a decreasing function of κ_2 [6] (this happens, for example, if $a < 0$). Later, Hopf proved in [14] that there are no closed analytic surfaces of genus greater or equal than 2 unless $a = -1$, that is, the surface has CMC and if the genus is 0 and the surface is analytic and rotational, then a or $1/a$ must be an odd integer. Indeed, for each $a > 1$, Hopf proved the existence of non-spherical closed convex rotational $\mathcal{C}^2$-surfaces.

Moreover, he also proved in [14] the existence of convex closed rotational surfaces for any $a > 0$. When $a = 2$ and $b = 0$, Mladenov and Oprea have named this surface as the Mylar balloon [17] and [18]. If $a > 0$ and $b = 0$, they have also given parametrizations of the closed surfaces in terms of elliptic and hypergeometric functions and show that the surface is a critical point of a variational problem [17]. For any a and $b = 0$, Barros and Garay proved that all the parallels of these rotational surfaces are critical points for an energy functional involving the normal curvature and acting on the space of closed curves immersed in the surface [4].

In this setting, one of the oldest variational problems over curves was to determine the shape of an ideal *elastic rod*, which goes back to 1691. Based on D. Bernoulli's approach, an *elastic curve* is a minimizer of the *bending energy*, that is, considering a regular curve γ with curvature κ, its bending energy is given by

$$\mathbf{\Theta}(\gamma) = \int_\gamma \kappa^2 \, ds \tag{2}$$

where s is the arc-length parameter of γ. In 1744, Euler published his classification of the planar elastic curves in the Euclidean plane, $\mathbb{R}^2$, [10].

In Section 2 we are going to analyze the critical curves of a generalized bending energy, which is going to be called *p-elastic energy* along the paper, following the terminology of [12]. Then, in Section 3, we are going to use the *binormal evolution procedure*, [2], to construct some invariant surfaces from these p-elastic curves. Indeed, we are going to evolve them under the naturally associated *Killing vector field* in the direction of the binormal with prescribed velocity. It turns out that the invariant surfaces constructed in this way are, precisely, rotational linear Weingarten surfaces [16]. Moreover, in Section 4, we are going to describe, locally, the profile curve of any rotational linear Weingarten surface as a planar p-elasticae [16]. Thus, combining Section 3 and Section 4, we obtain a characterization of rotational linear Weingarten surfaces in terms of planar p-elastic curves.

Finally, we particularize our findings to two remarkable cases. Firstly, we consider classic elastic curves, that is, critical curves of (2), and due to our construction, we recover the characterization of generating curves of Mylar balloons as elastic curves. Secondly, by considering critical curves of a Blaschke's variational problem (which we will show that they represent the roulettes of conic foci), we obtain all CMC rotational surfaces [3].

2. The p-Elastic Energy of Curves

Let's consider the Euclidean 3-space, $\mathbb{R}^3$, with metric $g \equiv \langle \cdot \, , \cdot \rangle$ and Levi-Civita connection ∇. If $\gamma : \bar{I} \to \mathbb{R}^3$ is a smooth immersed curve in $\mathbb{R}^3$, $\gamma'(t)$ will represent its velocity vector $\frac{d\gamma(t)}{dt}$ and the covariant derivative of a vector field $X(t)$ along γ will be denoted by $X'(t)$.

Any curve can be parametrized by the arc-length and this natural parameter is going to be denoted by s and, therefore, the tangent to the curve is going to be represented by $T(s) = \gamma'(s)$. The *first Frenet curvature*, or simply, *the curvature,* is defined as the positive root of $\kappa_1^2 = \langle T'(s), T'(s)\rangle$. A *geodesic* is a constant speed curve whose tangent vector is parallel propagated along itself, that is, a curve whose tangent, $T(s)$, satisfies the equation $T'(s) = 0$. Obviously, geodesics have zero curvature.

Let's denote by γ an arc-length parametrized curve immersed in $\mathbb{R}^3$, and let $\kappa(s)$ represent the curvature of γ. If $\kappa(s) = 0$, then γ is a geodesic in $\mathbb{R}^3$. On the other hand, if $\gamma(s)$ is a unit speed non-geodesic smooth curve immersed in $\mathbb{R}^3$, then $\gamma(s)$ is a *Frenet curve* of rank 2 or 3 and the standard *Frenet frame* along $\gamma(s)$ is given by $\{T, N, B\}(s)$, where N and B are the *unit normal* and *unit binormal* to the curve, respectively, and B is chosen so that $det(T, N, B) = 1$. Then the *Frenet equations*

$$T'(s) = \kappa(s)\, N(s) \tag{3}$$

$$N'(s) = -\,\kappa(s)\, T(s) + \tau(s)\, B(s) \tag{4}$$

$$B'(s) = -\,\tau(s)\, N(s) \tag{5}$$

define the *curvature*, $\kappa(s)$, and *torsion*, $\tau(s)$, along $\gamma(s)$.

Now, the following relations hold

$$T = N \times B\,, \quad N = B \times T\,, \quad B = T \times N\,.$$

Notice that, even if the rank of γ is 2 (that is, $\tau = 0$), the binormal $B = T \times N$ is still well defined and above formulas (3)-(5) still make sense when $\tau = 0$. A curve with vanishing torsion, $\tau = 0$, is going to be referred as a *planar curve*.

Now, let's consider for any $p, \mu \in \mathbb{R}$, the *curvature energy functional*

$$\mathbf{\Theta}_\mu^p(\gamma) = \int_o^L (\kappa(s) - \mu)^p \, ds\,, \tag{6}$$

where L denotes the length of γ. This functional has been studied in [12], where its critical curves have been called *p-elastic curves*. Therefore, along this paper, we are going to refer to $\mathbf{\Theta}_\mu^p$, (6), as the *p-elastic energy*.

Notice that the case $p = 0$ corresponds with the length functional. On the other hand, the case $p = 1$ is, basically, the total curvature functional, [1] (check also the references therein). From now on, we discard these two cases, so $p \neq 0, 1$. The curvature energy functional (6) generalizes more classical variational problems, for instance, if $\mu = 0$ and $p = 2$, we recover the bending energy (2) whose critical curves are usually called elastic curves. This is the reason why we call p-elastic curves to extremals of the generalized case (6). Finally, we recall that the particular case given by $\mu = 0$ and $p = 1/2$ was studied by Blaschke in [5], obtaining that the critical curves are catenaries.

It is clear form (6) that for any $p > 0$, curves with $\kappa = \mu$ will be global minima among all the curves with $(\kappa - \mu)^p \in L^1([0, L])$. Then, for all the cases, we consider $\mathbf{\Theta}_\mu^p(\gamma) \geq 0$ acting on the following space of planar curves satisfying given boundary conditions with κ greater than μ. We shall denote by $\Omega_{p_o p_1}$ to the space of smooth immersed Frenet curves of rank 2 joining two given points and verifying that $\kappa > \mu$, that is,

$$\Omega_{p_o p_1} = \{\beta : [0,1] \to \mathbb{R}^2 \; ; \; \beta(i) = p_i, i \in \{0,1\}, \beta'(t) \neq 0, \forall t \in [0,1], \kappa > \mu\},$$

where $p_i \in \mathbb{R}^2$, $i \in \{0, 1\}$ are arbitrary given points.

Then, by standard arguments involving integration by parts we obtain the *Euler-Lagrange equation* of $\mathbf{\Theta}_\mu^p$ acting on $\Omega_{p_o p_1}$,

$$\frac{d^2}{ds^2}\left((\kappa - \mu)^{p-1}\right) + \kappa^2(\kappa - \mu)^{p-1} - \frac{1}{p}\kappa(\kappa - \mu)^p = 0\,. \tag{7}$$

From now on, we will call *critical curve* (or, also, *extremal curve*) to any curve $\gamma \subset \Omega_{p_o p_1}$, whose curvature κ verifies the Euler-Lagrange equation (7).

Observe that above Euler-Lagrange equation, (7), can be directly integrated once. Moreover, after some manipulations we have the following expression for the derivative of the curvature

$$\left(\kappa'\right)^2 = \frac{(\kappa - \mu)^2}{p^2(p-1)^2}\left(d\,(\kappa - \mu)^{2(1-p)} - ((p-1)\kappa + \mu)^2\right),$$

where d is a positive real constant.

3. Binormal Evolution of p-Elasticae

Along this section we are going to consider that γ is any critical Frenet curve of rank 2 of $\mathbf{\Theta}_\mu^p$ acting on $\Omega_{p_o p_1}$, and we are going to evolve it under the associated Killing vector field in the direction of the binormal. For this purpose, we are going to consider that $\mathbb{R}^2$ is contained in $\mathbb{R}^3$.

A vector field W along γ, which infinitesimally preserves unit speed parametrization is said to be a *Killing vector field along* γ (in the sense of [15]) if γ evolves in the direction of W without changing shape, only position. In other words, the following equations must hold

$$W(v)(s,0) = W(\kappa)(s,0) = 0\,, \tag{8}$$

($v = |\gamma'|$ being the speed of γ) for any variation of γ having W as variation field.

It turns out that p-elastic curves of $\mathbf{\Theta}_\mu^p$ have naturally associated Killing vector fields defined along them. Let us define the following vector field, $\mathcal{I}$, along γ

$$\mathcal{I} = p\,(\kappa-\mu)^{p-1}\,B\,. \tag{9}$$

Now, as proved for instance in [11], combining both the Euler-Lagrange equation of a planar p-elastic curve, (7), and the definition of Killing vector fields along curves, (8), we obtain

Proposition 1. *Assume that γ is a planar p-elastic curve and consider the vector field $\mathcal{I}$,* (9)*, defined on γ. Then $\mathcal{I}$ is a Killing vector field along γ.*

Then, following the arguments of [15], this vector field along γ, $\mathcal{I}$, can be uniquely extended to the whole $\mathbb{R}^3$. We are going to use ξ to represent this extension and we are going to denote by $\{\phi_t\,;\ t\in\mathbb{R}\}$ the one-parameter group of isometries associated to ξ, that is, the flow of ξ.

Now, we locally define the *binormal evolution surface* (for more details see [2] and references therein)

$$S_\gamma := \{x(s,t)=\phi_t\left(\gamma(s)\right)\}\,.$$

Then, we have the following result

Proposition 2 ([3])**.** *Let γ be a critical curve of $\mathbf{\Theta}_\mu^p$ acting on $\Omega_{p_o p_1}$. Then, the binormal evolution surface with initial condition γ defined above is a rotational surface.*

Moreover, if γ is a critical curve of $\mathbf{\Theta}_\mu^p$ acting on $\Omega_{p_o p_1}$ with constant curvature, then S_γ defined as above is a flat isoparametric surface and, as a consequence, it trivially verifies the linear relation (1) for some suitable real constants a and b.

On the other hand, for the non-constant curvature case, we have

Theorem 3 ([16])**.** *Let γ be a critical curve of $\mathbf{\Theta}_\mu^p$ acting on $\Omega_{p_o p_1}$ with non-constant curvature. Then, the binormal evolution surface with initial condition γ, S_γ, is a rotational linear Weingarten surface where*

$$a=\frac{p}{p-1} \qquad \text{and} \qquad b=\frac{\mu}{p-1}\,.$$

4. Characterization of Rotational Linear Weingarten Surfaces

Notice that in previous section we have given a way of constructing rotational linear Weingarten surfaces by evolving planar p-elastic curves under the binormal flow with prescribed velocity $|\mathcal{I}|$, see (9).

Moreover, the converse is almost true. Indeed, in this section we are going to show that for any rotational linear Weingarten surfaces with $a\neq 1$ it is possible to find a local coordinate system, such that, the orthogonal curve to the rotation is a planar p-elasticae and, therefore, all rotational linear Weingarten surfaces with $a \neq 1$ can be locally described as binormal evolution surfaces with planar p-elastic curves as initial condition and with velocity given by $|\mathcal{I}|$, see (9).

Let S be a rotational linear Weingarten surface. Then, there exists a planar curve γ in $\mathbb{R}^3$ such that S can be seen as the set $\{\phi_t(\gamma(s))\ ;\ t \in \mathbb{R}\}$, where now $\{\phi_t\ ;\ t \in \mathbb{R}\}$ represents a one-parameter group of rotations. Therefore, along this section we are going to use the notation $S = S_\gamma$, and we are going to say that γ is the *profile curve*.

Let us assume first that γ is a geodesic of $\mathbb{R}^3$. Then, it is clear that γ is a global minima of $\mathbf{\Theta}_\mu^p$ since $\kappa = 0$, as explained in Section 2. Moreover, it can easily be checked that the rotational surface with profile curve γ, S_γ, is a flat ruled isoparametric surface.

Therefore, we assume now that γ is not a geodesic of $\mathbb{R}^3$, then we have

Theorem 4 ([16]). *Let S_γ be a rotational surface with profile curve γ and such that its principal curvatures verify the linear relation* (1) *for $a \neq 1$. Then, γ is an extremal curve of $\mathbf{\Theta}_\mu^p$ for*

$$\mu = \frac{b}{a-1} \qquad \textit{and} \qquad p = \frac{a}{a-1}\,.$$

Finally, for the sake of completeness, we summarize the complete description of rotational linear Weingarten surfaces and of their corresponding profile curves (see Figures 1 to 3). The complete geometric classification depending on the values of the constants a and b in the linear relation (1), can be found in [16], where it has been proved using a different tool.

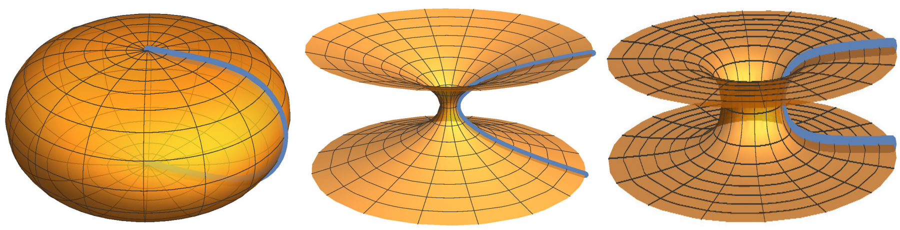

Figure 1. From left to right: Ovaloid ($a > 0$), Catenoid-Type Surface with $a \in [-1, 0)$ and Catenoid-Type Surface with $a < -1$.

If the surface does not meet the axis of rotation, we have the next type of surfaces;

- *Catenoid-Type Surfaces*. The profile curve γ is a concave graph on some interval I of the axis. These surfaces only appear when $a < 0$ and $b = 0$. There are two types depending if $I = \mathbb{R}$ ($-1 \leq a < 0$) or if I is a bounded interval ($a < -1$). The plane is included here as a limit case.

- *Unduloid-Type Surfaces*. Embedded surfaces which are periodic in the direction of the axis. Circular cylinders belong to this family.
- *Nodoid-Type Surfaces*. Non embedded surfaces which are periodic in the direction of the axis and the profile curve γ has loops towards the axis.
- *Antinodoid-Type Surfaces*. Non embedded surfaces which are periodic in the direction of the axis and the profile curve γ has loops facing away from the axis.
- *Cylindrical Antinodoid-Type Surfaces*. Non embedded surfaces asymptotic to a circular cylinder. The profile curve γ has a single loop facing away from the axis.

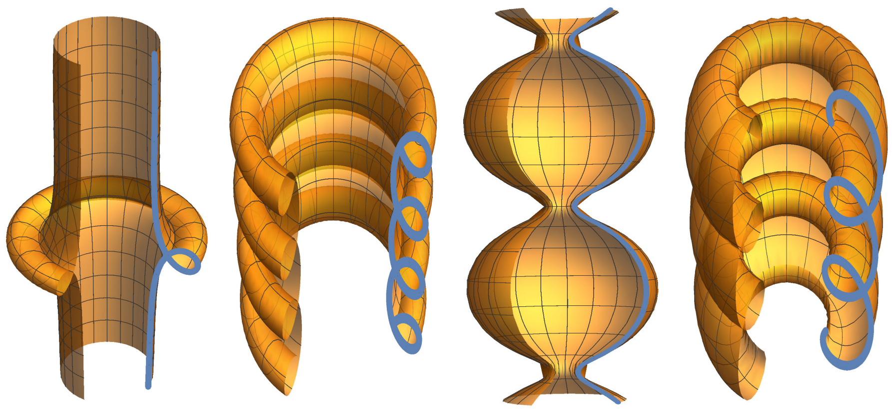

Figure 2. From left to right: Cylindrical Antinodoid-Type Surface, Antinodoid-Type Surface, Unduloid-Type Surface and Nodoid-Type Surface.

Then, we turn to those surfaces that meet (necessarily orthogonally) the axis of rotation. All the surfaces have genus 0 except in one case that the surface touches the axis at exactly one point (pinched spheroids).

- *Ovaloids*. They are convex surfaces. The shape is like an oblate spheroid being more flat close to the axis as the parameter a gets bigger. This case only occurs when $a > 0$. Round spheres are included here.
- *Vesicle-Type Surfaces*. Embedded closed surfaces where the two poles of the profile curve are close so the profile curve presents two inflection points. These surfaces have concave regions around the poles.
- *Pinched Spheroids*. Limit case of vesicle-type surfaces when the two poles coincide. The surface is tangentially immersed on the axis and bounds a solid three-dimensional torus.

- *Immersed Spheroids*. Closed surfaces of genus 0 that appear when the two poles of the vesicle-type surface pass their-self through the axis.

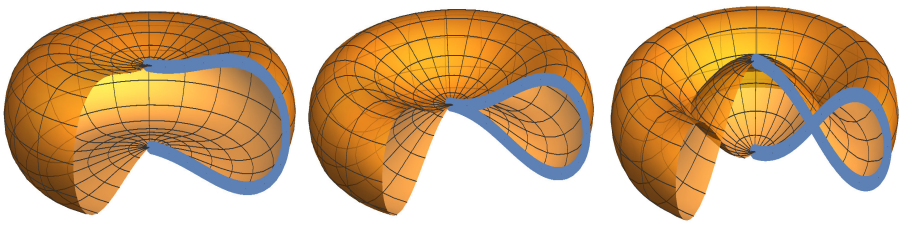

Figure 3. From left to right: Vesicle-Type Surface, Pinched Spheroid and Immersed Spheroid.

5. Application to Two Remarkable Families

To end up this article, we are going to illustrate this characterization by applying our findings to two well-known classical families of surfaces, Mylar balloons and Delaunay surfaces.

5.1. Mylar Balloons

As mentioned in the introduction, one of the oldest variational problems over curves was to find the critical curves of the bending energy, (2). These curves are usually called *elasticae* and they were described by Euler in [10]. Later, it was discovered that, by using the Jacobi cosine, cn, elastic curves could be either geodesics or characterized, up to rigid motions, by the following family of curvatures

$$\kappa(s) = \kappa_o \, cn \left(\frac{\kappa_o}{\sqrt{2}} \, s \, , \, \frac{\sqrt{2}}{2} \right) ,$$

where κ_o is a constant representing the *maximum curvature*. Following the notation of previous sections, this constant, κ_o, is related with our constant of integration, d, so that the first integral of the Euler-Lagrange equation is verified.

On the other hand, in 1906 Da Rios, [7], modeled the self-induced movement of a thin vortex filament in a viscous fluid traveling without stretching. For this model, Da Rios used the motion of a curve, which represents the filament, propagating according to the localized induction equation. Then, Hasimoto related this motion with elastic curves proving that the evolution of the curve is done under the binormal flow, [13]. Therefore, the binormal evolution surfaces with elastic curves as initial conditions are called *Hasimoto surfaces*.

In our particular case, combining the results of Theorem 3 and Theorem 4 we obtain

Theorem 5. *The binormal evolution surface generated from a planar elastic curve is a rotational Hasimoto surface verifying that*

$$\kappa_1 = 2\,\kappa_2$$

κ_i*,* $i \in \{1, 2\}$ *denoting the principal curvatures. Furthermore, the converse is also true.*

Observe that the rotational surfaces verifying the relation, $\kappa_1 = 2\,\kappa_2$, between their principal curvatures are essentially unique, up to translations and homotheties, as pointed out in [17] (see also [16]). Moreover, in [17], Mladenov and Oprea named *Mylar balloon* to this surface. The term Mylar balloon was first introduced by Paulsen, [19], to denote the resulting object of a physical experiment when trying to understand the shape of the surface made by Mylar (a non-stretchable material) that encloses the maximum volume, (see Figure 1 on the left).

Therefore, as a consequence of Theorem 5, we recover the following result

Corollary 6. *The profile curve of the Mylar balloon is a planar elastic curve.*

In the literature, this result is attributed to Gibbons (see, for instance, [9] and references therein).

5.2. Delaunay Surfaces

We consider now the particular p-elastic energy given by the choice $p = 1/2$. That is,

$$\mathbf{\Theta}_\mu^{1/2}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds\,. \tag{10}$$

If $\kappa = \mu$, then γ is an absolute minima provided we are considering $L^1([0, L])$ as the space of curves, see Section 2. On the other hand, if $\mathbf{\Theta}_\mu^{1/2}$ is acting on $\Omega_{p_o p_1}$, then by solving the Euler-Lagrange equation we get the curvature of the critical curves. We distinguish two cases; if the curvature is constant, then γ is a curve verifying $\kappa = 2\mu$. On the other hand, for the non-constant curvature case we obtain for every $d > 0$ (see [3])

$$\kappa(s) = \frac{4d}{1 + 16d^2 s^2}\,,$$

if $\mu = 0$. And, if $\mu \neq 0$,

$$\kappa(s) = \frac{2\mu\left(w^2 + w\sin(2\mu s)\right)}{1 + w^2 + 2w\sin(2\mu s)}\,,$$

where $w^2 = 1 + \mu/d$.

Therefore, the following result can be computed directly, since the curvature of planar curves completely determines them, up to rigid motions.

Proposition 7 ([3]). *Critical curves of the extended Blaschke's variational problem, $\mathbf{\Theta}_\mu^{1/2}$, are precisely the roulettes of conic foci.*

Indeed, if the curvature is constant, then the critical curve may be a *line* or a *circle*, depending if the parameter μ is zero or not. Moreover, for the other cases we get a *catenary* for $\mu = 0$ (recovering the result of Blaschke, [5]), a *nodary* for $\mu \neq 0$ and $w < 1$ and an *undulary* for $\mu \neq 0$ and $w > 1$.

Furthermore, this property serves as a geometric characterization of critical curves of $\mathbf{\Theta}_\mu^{1/2}$ as roulettes of conics. In fact, the loci of a focus of a conic as the point of contact rolls along a straight line without slipping in a plane are *roulettes of conics* in $\mathbb{R}^2$. A line is generated when the conic is a *circle*, since its focus coincides with its center. If the conic is a *parabola*, its focus traces a catenary. For a *hyperbola*, we get a nodary; for a *proper ellipse*, an undulary; and, finally, if our conic is degenerate we obtain a circle.

Notice that already in 1841, Delaunay introduced a way of constructing rotational CMC surfaces in $\mathbb{R}^3$, by proving that, basically, a *rotational surface in* $\mathbb{R}^3$ *is a CMC surface, if and only if, its profile curve is the roulette of a conic*, [8]. Therefore, we conclude with

Theorem 8 ([3]). *A Delaunay surface is, precisely, a binormal evolution surface with a critical curve for the extended Blaschke's energy, $\mathbf{\Theta}_\mu^{1/2}$, as initial condition. Moreover, the CMC is $H = -\mu$.*

This result can be proved directly from our variational characterization introduced in previous sections. For this proof and more detailed explanations see [3].

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