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Universidad del País Vasco / Euskal Herriko Unibertsitatea
ZTF-FCT
Zientzia eta Teknologia Fakultatea
Facultad de Ciencia y Tecnología

Invariant Surfaces with Generalized Elastic Profile Curves

Álvaro Pámpano Llarena

Doctoral Thesis

Advisors:
Dr. Óscar J. Garay Bengoechea
Dr. Josu Arroyo Olea

Main Objective and Scheme

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Study the connection between generalized elastic curves and invariant surfaces possessing nice geometric properties.

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    - Curvature Energies
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   - Application to visual curve completion

Curvature Energies (with Potential)

Consider the following curvature energy functional for a potential Φ,

$$\boldsymbol{\Theta}(\gamma) = \int_\gamma (P(\kappa) + \Phi)\, ds$$

acting on a space of immersed Frenet curves of M_r^n.

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$$\begin{aligned}\mathcal{E}(\gamma) \;&=\; \widetilde{\nabla}_T\left(\widetilde{\nabla}_T(\dot{P}N) + \varepsilon_1(2\kappa\dot{P} - P - \Phi)T\right) \\ &\quad + \dot{P}\, R(N,T)T + \operatorname{grad}\Phi\,,\end{aligned}$$

where $\dot{P}$ denotes the derivative of P with respecto to κ.

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Convention

We are going to call critical curve or extremal curve to any Frenet curve of M_r^n verifying $\mathcal{E}(\gamma) = 0$.

Different Types of Critical Curves

- If $P(\kappa) = \kappa^2$ and $M_r^n = M^2$, critical curves are elasticae with potential. And, the Euler-Lagrange equation boils down to

$$2\kappa_{ss} + \kappa\left(\kappa^2 + 2K - \Phi\right) + N(\phi) = 0\,.$$

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Euler-Lagrange Equations

$$\dot{P}_{ss} + \varepsilon_1\varepsilon_2\dot{P}\left(\kappa^2 - \varepsilon_1\varepsilon_3\tau^2 + \varepsilon_2\rho\right) - \varepsilon_1\varepsilon_2\kappa\left(P - \mu\,\tau + \lambda\right) = 0\,,$$

$$2\tau\dot{P}_s + \tau_s\dot{P} - \varepsilon_1\varepsilon_3\mu\,\kappa_s = 0\,.$$

The ε_i denotes the causal characters of the Frenet frame $\{T, N, B\}$.

Associated Killing Vector Fields

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A vector field W along a critical curve γ verifying

$$W(v)(s,0) = W(\kappa)(s,0) = W(\tau)(s,0) = 0$$

is a Killing vector field along γ. **([16]: Langer & Singer, 1984)**

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Proposition 1.3.3 **([31]: Garay & — , 2016)**

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Proposition 1.3.2 **([11]: Ferrández, Guerrero, Javaloyes & Lucas, 2016)**

The vector field $\mathcal{I} = \dot{P}\, B$ is a Killing vector field along γ, if and only if, γ is an extremal of

$$\mathbf{\Theta}(\gamma) = \int_\gamma (P(\kappa) + \lambda)\, ds.$$

Generalized Elastic Curves

In this memory we have considered the curvature energy functional

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Associated Killing Vector Field

In any semi-Riemannian 3-space form, $M_r^3(\rho)$, critical curves of $\mathbf{\Theta}_{\mu}^{\epsilon,p}$, have a naturally associated Killing vector field defined by

$$\mathcal{I} = \varepsilon \, p \, \kappa^{\epsilon-1} \, (\kappa^\epsilon - \mu)^{p-1} \, B.$$

And it extends to a Killing vector field in the whole $M_r^3(\rho)$.

Visual Curve Completion

For some applications see **([36]: Arroyo, Garay & — , submitted)**.

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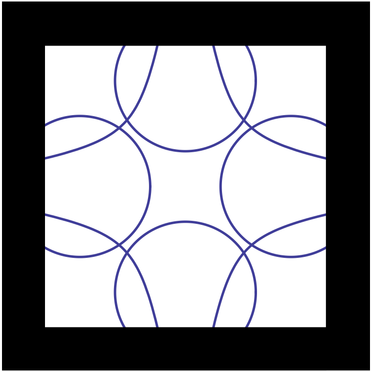
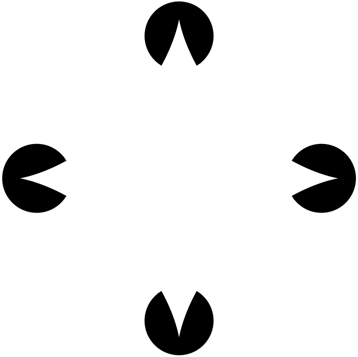
Problem: How to recover a covered or damaged image?





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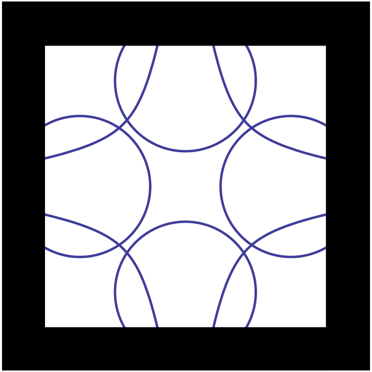

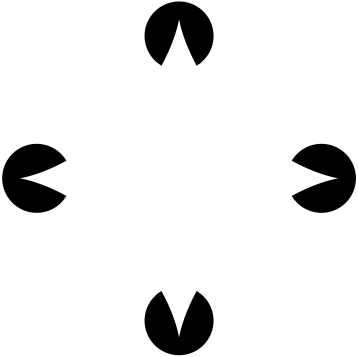

In our brain, the primary visual cortex, $V1$, gives us an intuitive answer.

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- Here, we consider the length functional.

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Geodesics in M^3 are obtained by lifting minimizers (more generally, critical curves) in $\mathbb{R}^2$ of

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- We completely solve the variational problem, geometrically.

([29]: Arroyo, Garay & — , 2015)

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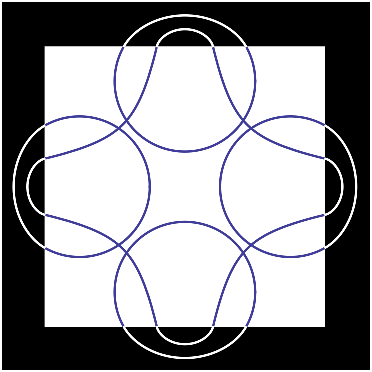

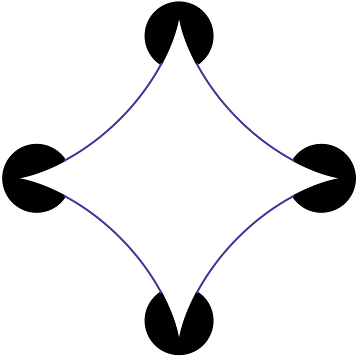

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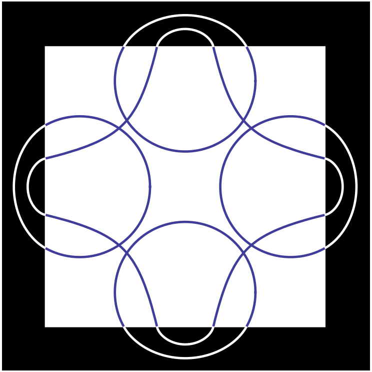

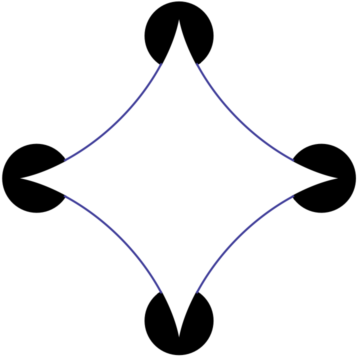

XEL-Platform (www.ikergeometry.org)

A gradient descent method useful for families of functionals defined on certain spaces of curves satisfying both affine and isoperimetric constraints. **([42]: Arroyo, Garay, Mencía & — , preprint)**

Part II

Binormal Evolution Surfaces

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1. Chapter 3. Binormal Evolution Surfaces in 3-Space Forms

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The Gauss-Codazzi equations for these evolutions are given by

$$\begin{aligned}\kappa_t &= -2\mathcal{F}_s\tau-\tau_s\mathcal{F}\,,\\ \tau_t &= \varepsilon_1\varepsilon_3\kappa\mathcal{F}_s+\varepsilon_2\left(\frac{\mathcal{F}}{\kappa}\left(\varepsilon_3\frac{\mathcal{F}_{ss}}{\mathcal{F}}-\varepsilon_2\tau^2+\varepsilon_1\varepsilon_3\rho\right)\right)_s\,.\end{aligned}$$

Traveling Wave Solutions

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A function $u(s,t)$ of the form $u(s,t)=\psi(s-\varpi\, t)$ with $\varpi\in\mathbb{R}$ is said to be a traveling wave.

Traveling Wave Solutions

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Traveling wave solutions of Gauss-Codazzi equations correspond to the Euler-Lagrange equations of generalized Kirchhoff centerlines.

Moreover, they evolve under the binormal flow by isometries and slippage.
In particular,

- **Corollary 3.1.4.** **([31])** A Frenet curve evolves under the binormal flow by isometries, if and only if, it is an extremal of

$$\mathbf{\Theta}(\gamma)=\int_\gamma \left(P(\kappa)+\lambda\right)\, ds\,,$$

where $\mathcal{F}(\kappa,\tau)=\dot{P}(\kappa)$.

Evolution with $\tau = 0$

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Proposition 3.2.1. ([32]: Arroyo, Garay & —, 2017)

If the initial filament $\gamma(s) = x(s,0)$ is planar, then it is an extremal curve for

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- **Proposition 3.2.3.** **([34])** If γ has constant curvature, then S_γ is a flat isoparametric surface.
- **Proposition 3.2.4.** **([34])** For general curvature, if $S_\gamma \subset M^3(\rho)$, then it is a rotational surface.

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2. If $d = 0$, δ_{s_o} is an horocycle and S_γ is a parabolic rotational surface.
3. If $d < 0$, δ_{s_o} is an hypercycle and S_γ is a hyperbolic rotational surface.

Closure Conditions in $M^3(\rho)$

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Then, there exists a one-parameter group of isometries of $M_r^3(\rho)$ such that a suitable parametrization of the surface S_γ is a solution of the binormal flow with $\mathcal{F}(\kappa(s,t)) = \frac{\varepsilon_1\varepsilon_3\mu}{2\tau_o}\,\kappa + \lambda$.

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- In this case, $\rho = (-1)^r \tau_o^2$, that is we are in $\mathbb{S}^3(\rho)$ or $\mathbb{H}_1^3(\rho)$. (Assume $\tau_o = 1$)

Proposition 3.4.1. **([32]: Arroyo, Garay & — , 2017)**

The corresponding binormal evolution surface evolving under $x_t = B$ by rigid motions is a Hopf cylinder of $\mathbb{S}^3(1)$ or $\mathbb{H}_1^3(-1)$.

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Gauss-Codazzi Equations **([28]: Garay, — & Woo, 2015)**

$$\begin{aligned} \kappa_t &= -\varepsilon_2\varepsilon_3\left(2\kappa_s\tau + \kappa\tau_s\right) \\ \tau_t &= \varepsilon_2\left(\varepsilon_2\frac{\kappa_{ss}}{\kappa} - \varepsilon_3\tau^2 + \frac{1}{2}\varepsilon_1\kappa^2 + \varepsilon_1\varepsilon_2\rho\right)_s \end{aligned}$$

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- They describe the movement of a vortex filament according to the localized induction equation.

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- Via the Hasimoto transformation, we get both the focusing and the defocusing nonlinear Schrodinger equation. **([13]: Hasimoto, 1972)**
- Finally, traveling wave solutions correspond with centerlines of Kirchhoff elastic rods.

Part III (Chapter 4)

Invariant Surfaces in 3-Space Forms

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1. Chapter 4. Invariant Constant Mean Curvature Surfaces

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- Delaunay Surfaces in Riemannian 3-Space Forms

Characterization as BES

Theorem 4.1.1. **([34]: Arroyo, Garay & — , 2018)**

Let S_γ be an invariant CMC surface of $M_r^3(\rho)$. Then, locally, S_γ is either a ruled surface

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Idea of the proof:

- Take a geodesic coordinate system in S_γ.
- Observe that solutions of the corresponding Gauss-Codazzi equations imply criticality of γ.

Ermakov-Milne-Pinney Equation

Notice that the velocity of previous binormal evolution surface is given by

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Theorem 4.1.2. **([34]: Arroyo, Garay & — , 2018)**

The warping function $G(s)$ is a solution of the following EMP equation

$$G''(s)+\alpha\,G(s)=\frac{\varpi}{G^3(s)}\,.$$

Blaschke's Curvature Type Energy

For a fixed $\mu \in \mathbb{R}$, consider the curvature energy functional

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Corollary 4.2.4. ([31]: Garay & — , 2016)

In $\mathbb{R}^2$, critical curves of $\mathbf{\Theta}_\mu$ are the roulettes of conic foci.

Corollary 4.2.5. ([33]: — , 2017)

In $\mathbb{L}^2$, the locus of the origin when a part of a spacelike quadratic curve is rolled along a spacelike line is a spacelike critical curve.

Binormal Evolution of These Extremals

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Theorem 4.2.6. **([34]: Arroyo, Garay & — , 2018)**

The binormal evolution surface S_γ has CMC $H = -\varepsilon_1\varepsilon_2\,\mu$.

Binormal Evolution of These Extremals

Take γ any critical curve of $\mathbf{\Theta}_\mu$.

1. Consider the one-parameter group of isometries determined by the flow of the extension of

$$\mathcal{I} = \frac{1}{2\sqrt{\kappa - \mu}}\, B\,.$$

   Since $M_r^3(\rho)$ is complete we can denote it by $\{\phi_t\,,\ t \in \mathbb{R}\}$.

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In conclusion, CMC invariant surfaces of $M_r^3(\rho)$ are, locally, either

- Ruled surfaces S_γ (γ being a geodesic), or
- Surfaces S_γ swept out by extremals γ of $\mathbf{\Theta}_\mu$.

Bour's Families

In particular, when γ has non-constant curvature, we have a two-parameter family of invariant surfaces in $M_r^3(\rho)$, $\mathcal{F}_{d,e}$, with the same CMC $H = -\varepsilon_1\varepsilon_2\mu$.

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Consider the conic $\mathcal{C}_\nu$ in the (d, e)-plane defined by

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1. $d \neq -\varepsilon_1\frac{\mu}{2}$ and $1 + \varepsilon_1\varepsilon_2 a\nu > 0$.
2. $\varepsilon_1\rho < -\varepsilon_1\varepsilon_2\mu^2$, $d \neq -\varepsilon_1\frac{\mu}{2}$ and $1 + \varepsilon_1\varepsilon_2 a\nu < 0$.

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Then, the family $\mathcal{F}_{d,e} \equiv \mathcal{F}_d^\nu$ represents a one-parameter isometric deformation of invariant surfaces with the same CMC.

- Moreover, for the $\kappa(s) = \kappa_o$ case, we obtain "limit" surfaces of the family $\mathcal{F}_d^\nu$.

Lawson's Type Correspondence

There exists a correspondence between CMC surfaces in different 3-space forms.

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1. $M_r^3(\rho)$ with CMC $|H| = |\mu|$,
2. $M_r^3(\widehat{\rho})$ with CMC $|\widehat{H}| = |\widehat{\mu}|$.

Rotational CMC Surfaces in $M^3(\rho)$

- A complete immersed CMC surface in $\mathbb{R}^3$ is helicoidal, if and only if, it is in the Bour's family of a rotational CMC surface.
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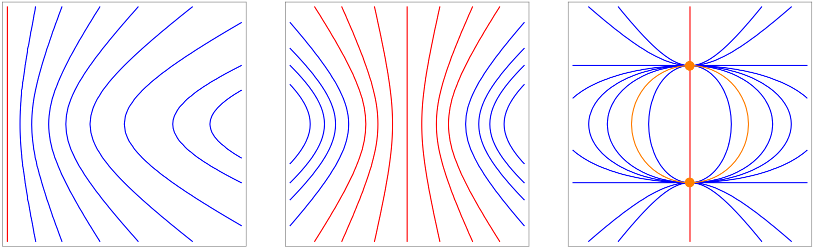

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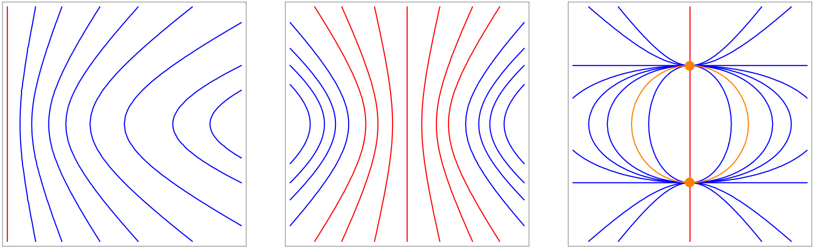

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Theorem 4.2.9. **([34]: Arroyo, Garay & — , 2018)**

All CMC invariant surfaces of Riemannian 3-space forms can be isometrically deformed into rotational surfaces with the same CMC.

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- Moreover, a similar result is true in $\mathbb{H}^3(\rho)$. **([1]: Aledo & Gálvez, 2002)**
- This suggests to study Delaunay surfaces in $\mathbb{S}^3(\rho)$.

Local Classification in $\mathbb{S}^3(\rho)$

Theorem 4.3.3. ([40]: **Arroyo, Garay & —** , ***submitted***; **and,** [41]: **Arroyo, Garay & —** , ***preprint***)

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$$\mathbb{S}^1\left(\sqrt{\rho+\kappa^2}\right) \times \mathbb{S}^1\left(\frac{\sqrt{\rho}}{\kappa}\sqrt{\rho+\kappa^2}\right)$$

if $\kappa(s) = -|H| + \sqrt{H^2+\rho}$.

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4. A binormal evolution surface where γ is a planar non-constant curvature critical curve of $\mathbf{\Theta}_\mu$ for $|\mu| = |H|$.

Critical Curves of Θ_μ in $\mathbb{S}^2(\rho)$

- Surfaces S_γ of point 4 in Theorem 4.3.3 depend greatly on γ.

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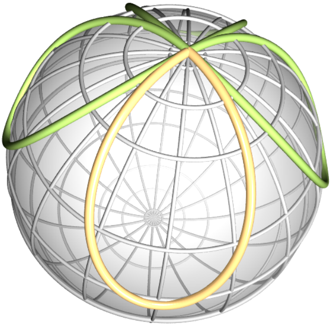
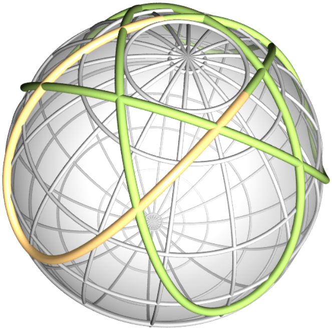


$\mu \simeq 0.312$ and $4\mu d = 1$



$\mu = -0.1$ and $d \simeq 1.27$

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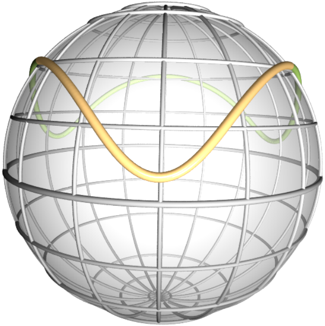
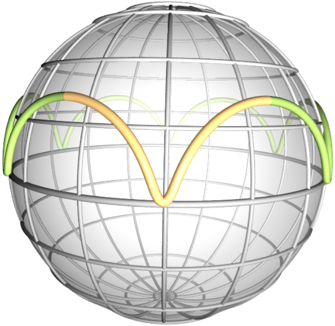


$\mu = -1$ and $d \simeq 2.48$



$\mu = -2$ and $d \simeq 16.19$

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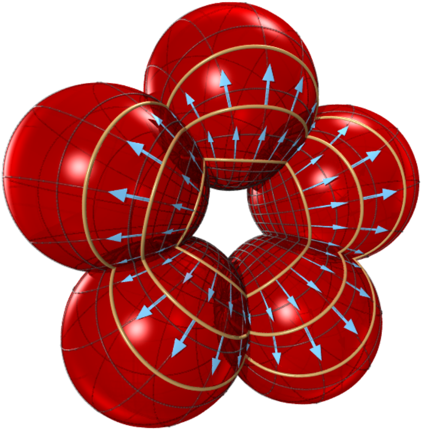

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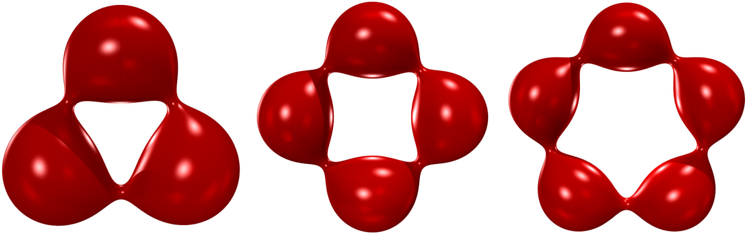

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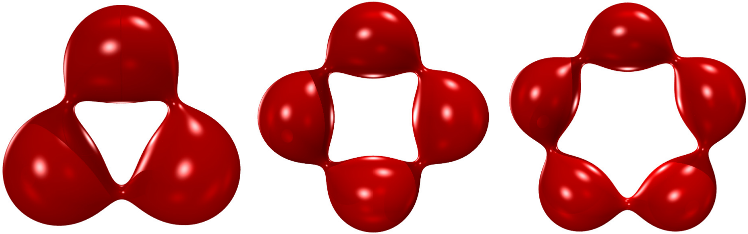

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- **([22]: Perdomo, 2010)** For any $m > 1$ and any H such that

$$|H| \in \left(\sqrt{\rho} \cot \frac{\pi}{m} , \sqrt{\rho} \frac{m^2 - 2}{2\sqrt{m^2 - 1}} \right)$$

exists a non-isoparametric embedded CMC rotational tori.

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Lawson's Conjecture (**[18]: Lawson, 1970**)

The only embedded minimal tori in $\mathbb{S}^3(\rho)$ is the Clifford torus. (Recently proved in **([7]: Brendle, 2013)**)

Part III (Chapter 5)

Invariant Surfaces in 3-Space Forms

1. Chapter 5. Invariant Linear Weingarten Surfaces Surfaces

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- The simplest relation extending the CMC condition is

$$\kappa_1 = a\,\kappa_2 + b.$$

Linear Weingarten Surfaces

In Riemannian backgrounds the mean curvature H can be written as

$$2H = \kappa_1 + \kappa_2$$

where κ_i are the principal curvatures.

- The CMC condition gives rise to a relation of the type $\Upsilon(\kappa_1, \kappa_2) = 0$.
- A surface verifying a certain relation $\Upsilon(\kappa_1, \kappa_2) = 0$ is called a Weingarten surface. **([27]: Weingarten, 1861)**
- The simplest relation extending the CMC condition is

$$\kappa_1 = a\,\kappa_2 + b.$$

Linear Weingarten Surfaces

A surface in $M^3(\rho)$ verifying $\kappa_1 = a\,\kappa_2 + b$ is a linear Weingarten surface.

The p-Elastic Energy in $M^3(\rho)$

For any fixed $\mu \in \mathbb{R}$, the p-elastic energy of curves is defined by **([37]: Garay & — , submitted)**

$$\mathbf{\Theta}_\mu^p(\gamma) = \int_\gamma (\kappa - \mu)^p \, ds.$$

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- If $p = 1/2$, we obtain Blaschke's type energy.

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Theorem 5.1.1. **(Extension of [37]: Garay & — , *submitted*)**

Assume that γ is a p-elastic curve, then, the function $\zeta(s)$ is a solution of

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- **Theorem 5.1.2.** **(Extension of [37])** Conversely, given any solution $\zeta(s)$ we can construct a critical curve of $\mathbf{\Theta}_\mu^p$.
- The curvature is given by above formula, while the torsion comes from

$$\zeta^4(s)\,\tau^2 + \varpi = 0.$$

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Theorem 5.2.1. **(Extension of [38]: López & — , *submitted*)**

Let S_γ be a rotational surface verifying

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with $a \neq 1$, then γ is a planar p-elasticae.

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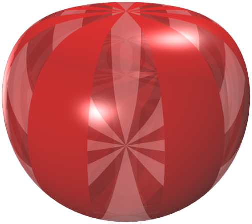
Theorem 5.2.4. ([4]: Barros & Garay, 2012)

The rotational linear Weingarten surfaces satisfying the relation $\kappa_1 = a\,\kappa_2$, $a \neq 0$, are ovaloids, catenoid-type surfaces and planes.



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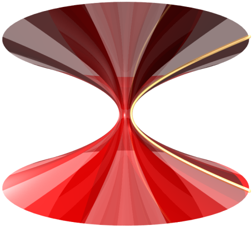

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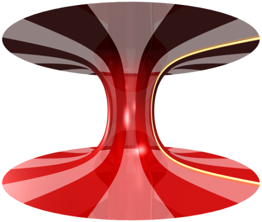

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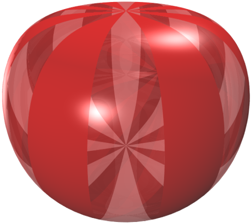

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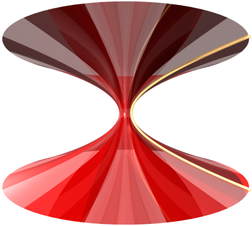

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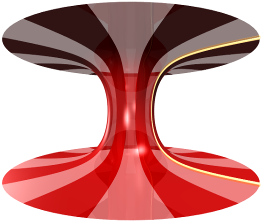

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- In particular, when $a = 2$ and $b = 0$, it is a Mylar balloon. **([20]: Mladenov & Oprea, 2003)**

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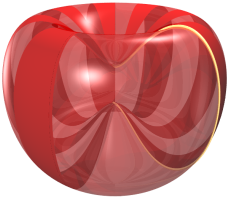
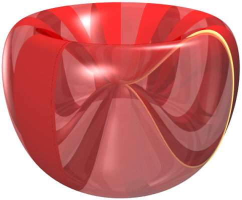
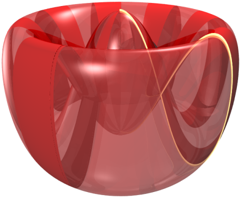

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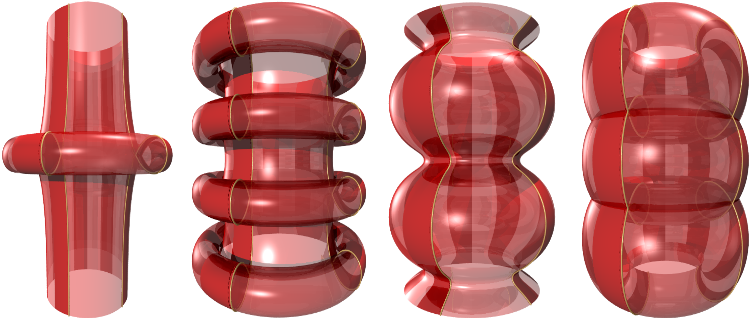

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An isometric immersion is biconservative if the corresponding stress-energy tensor is conservative.

- A biconservative surface is either a CMC surface or a rotational surface. **([8]: Caddeo, Montaldo, Oniciuc & Piu, 2014)**

Characterization as BES

- **Proposition 5.3.1.** **([43])** Non-CMC biconservative surfaces are rotational linear Weingarten surfaces for

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Theorem 5.3.2 & 5.3.3. **([43]: Montaldo & — , *preprint*)**

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Now, using closure conditions we have

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Killing Submersions

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They are the canonical models with constant K_B and τ_π.

- They include all 3-dimensional homogeneous spaces with group of isometries of dimension 4.

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Let γ be an immersed curve in B.

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- In fact, all ξ-invariant surfaces of M can be seen as vertical lifts of curves.
- The mean curvature of these surfaces is **([3]: Barros, 1997)**

$$H = \frac{1}{2}\left(\kappa \circ \pi\right) ,$$

κ denoting the geodesic curvature of γ in B.

Willmore-Like Surfaces in Total Spaces

Let $\Phi \in \mathcal{C}^\infty(M)$ be an invariant potential, that is, $\Phi = \bar{\Phi} \circ \pi$, and consider the Willmore-like energy

$$\mathcal{W}_\Phi(N^2) = \int_{N^2} \left(H^2 + \Phi\right) \, dA$$

defined on the space of surface immersions in a total space of a Killing submersion with compact fibers, $Imm(N^2, M)$.

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Theorem 6.3.1. **([35]: Barros, Garay & — , 2018)**

If γ is a closed curve in B, then S_γ is a Willmore-like torus, if and only if, γ is an extremal of

$$\mathbf{\Theta}_{4\bar{\Phi}}(\gamma) = \int_\gamma \left(\kappa^2 + 4\bar{\Phi}\right) \, ds.$$

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Now, for $\phi \in Imm(N^2, M)$, we consider the Chen-Willmore energy

$$\mathcal{CW}(N^2) = \int_{N^2} \left(H_\phi^2 + R\right) \, dA_\phi$$

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Theorem 6.3.3. **([35]: Barros, Garay & — , 2018)**

A vertical torus S_γ is Willmore in M, if and only if, it is extremal of

$$\mathcal{W}_{\tau_\pi^2}(N^2) = \int_{N^2} \left(H^2 + \tau_\pi^2\right)\, dA.$$

- That is, if and only if, γ is an elastica with potential $4\tau_\pi^2$ in B.

Willmore Tori Foliations of Total Spaces

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- Consider $S_f = I \times_f \mathbb{S}^1$ such that all fibers, δ, are extremals of

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- These S_f give rise to orthonormal frame bundles admitting foliations by Willmore tori with CMC.

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- **Corollary 6.4.4. ([35])** There exists a Killing submersion admitting a foliation by Willmore tori with CMC.

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