# Ground State Equilibria for the Helfrich Energy with Elastic Boundary

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**Abstract**

*We discuss ground state equilibrium surfaces for an energy which is a linear combination of the classical bending energy for curves and a surface energy containing the squared $L^2$ norm of the difference of the mean curvature and the spontaneous curvature, i.e. the Helfrich energy. We focus on the case of topological discs. The results of this poster are contained in [5], in which other results related to equilibrium and minimizing configurations for both topological discs and annuli are also described.*

## 1. Historical Background

In 1691, J. Bernoulli formulated the problem of determining the shape of ideal elastic rods bent by forces and momenta acting at its ends alone.
Later on, in a letter to L. Euler, D. Bernoulli (a nephew of J. Bernoulli) suggested that these rods should bend along the curve which minimizes the potential energy of the strain, i.e. elastic curves.
More generally, following this model, an **elastic curve** is a critical point of the **bending energy**

$$\mathcal{E}[C] := \int_C \left(\kappa^2 + \lambda\right) ds\,,$$

where $\kappa$ denotes the curvature of the curve $C$.
Using this new-found variational formulation of elastic curves, L. Euler in 1744 classified and described all the possible qualitative types for untwisted planar rod configurations, [2].
In the first half of the 19th century, the works of S. Germain and S. Poisson proposed the 2-dimensional analogous of $\mathcal{E}$ to study the physical system associated with an elastic plate.
For an immersion $X:\Sigma\to\mathbb{R}^3$ of a surface $\Sigma$ the bending energy is given by

$$\mathcal{W}[X] := \int_\Sigma H^2\, d\Sigma\,,$$

where $H$ is the mean curvature.
This energy was studied by W. Blaschke in the early 20th century and later reintroduced by T. Willmore. Since then, it is usually referred to as the Willmore energy.
Beyond its mathematical interest, W. Helfrich proposed an extension of $\mathcal{W}$ based on liquid crystallography to model cellular membranes, [4].
The **Helfrich energy** has the general form

$$\mathcal{H}[X] := \int_\Sigma \left(a\left[H+c_o\right]^2 + bK\right) d\Sigma\,,$$

where $a$, $c_o$ and $b$ are constants motivated by the physics and $K$ is the Gaussian curvature.
Recently, for compact surfaces with boundary, combinations of surface and boundary energies have also been considered: the Euler-Plateau problem ([3]), the Kirchhoff-Plateau problem ([1]), the Euler-Helfrich problem ([5]),...

## 2. The Euler-Helfrich Variational Problem

Let $\Sigma$ be a compact, connected surface with boundary and $X:\Sigma\to\mathbb{R}^3$ an embedding. We also assume that $X(\Sigma)$ is an oriented surface of class $\mathcal{C}^4$.
The **Euler-Helfrich energy** for $X:\Sigma\to\mathbb{R}^3$ is the functional

$$E[X] := \int_\Sigma \left(a\left[H+c_o\right]^2 + bK\right) d\Sigma + \oint_{\partial\Sigma}\left(\alpha\kappa^2+\beta\right) ds\,,$$

where $a>0$, $\alpha>0$ and $c_o$, $b$ and $\beta$ are any real constants. The energy $E$ combines the Helfrich energy $\mathcal{H}$ on the interior of the surface with the bending energy $\mathcal{E}$ of the boundary.

**Proposition 1 (Rescaling)** *Let $X:\Sigma\to\mathbb{R}^3$ be critical for $E$. Then,*

$$2ac_o\int_\Sigma \left(H+c_o\right) d\Sigma + \beta\mathcal{L}[\partial\Sigma] = \alpha\oint_{\partial\Sigma}\kappa^2\, ds\,,$$

*where $\mathcal{L}$ denotes the length functional. In particular, if $H+c_o\equiv 0$ on $\Sigma$, then $\beta>0$ holds.*

For an equilibrium surface the following **Euler-Lagrange equations** hold:

$$\begin{aligned} \Delta H + 2\left(H+c_o\right)\left(H\left[H-c_o\right]-K\right) &= 0\,, && \text{on } \Sigma\,,\\ a\left(H+c_o\right)+b\kappa_n &= 0\,, && \text{on } \partial\Sigma\,,\\ J'\cdot\nu - a\partial_n H + b\tau_g' &= 0\,, && \text{on } \partial\Sigma\,,\\ J'\cdot n + a\left(H+c_o\right)^2 + bK &= 0\,, && \text{on } \partial\Sigma\,, \end{aligned}$$

where $\{T,\nu,n\}$ is the Darboux frame, $\kappa_n$ and $\tau_g$ are the normal curvature and geodesic torsion of the boundary, respectively, and the vector field $J$ is defined as

$$J := 2\alpha T'' + \left(3\alpha\kappa^2-\beta\right)T\,,$$

along $\partial\Sigma$.

## 3. Ground State Equilibria

We consider the **ground state** $H+c_o\equiv 0$ on $\Sigma$. Then, the Euler-Lagrange equations reduce to

$$\begin{aligned} b\kappa_n &= 0\,, && \text{on } \partial\Sigma\,,\\ J'\cdot\nu + b\tau_g' &= 0\,, && \text{on } \partial\Sigma\,,\\ J'\cdot n - b\tau_g^2 &= 0\,, && \text{on } \partial\Sigma\,. \end{aligned}$$

**Theorem 2 (Elastic Curves Circular at Rest)** *Let $X:\Sigma\to\mathbb{R}^3$ be an equilibrium with $H+c_o\equiv 0$. Then, each boundary component $C$ is a simple and closed critical curve for*

$$F[C] := \int_C \left(\left[\kappa+\mu\right]^2+\lambda\right) ds\,,$$

*where $\mu := \pm b/(2\alpha)$ and $\lambda := \beta/\alpha - \mu^2$.*

In what follows, we will assume that $\Sigma$ is a topological disc, so that it only has one boundary component $C$. The classification of ground state equilibrium configurations is given in the following result.

**Theorem 3 (Disc Type Critical Surfaces)** *Let $X:\Sigma\to\mathbb{R}^3$ be a constant mean curvature $H=-c_o$ disc type surface critical for $E$. Then:*

*(i) Case $b\neq 0$. The surface is a planar disc bounded by a circle of radius $\sqrt{\alpha/\beta}$ and $c_o=0$.*

*(ii) Case $b=0$. The boundary is either a circle of radius $\sqrt{\alpha/\beta}$ or it is a simple and closed (classical) elastic curve representing a torus knot of type $G(q,1)$ for $q>2$.*

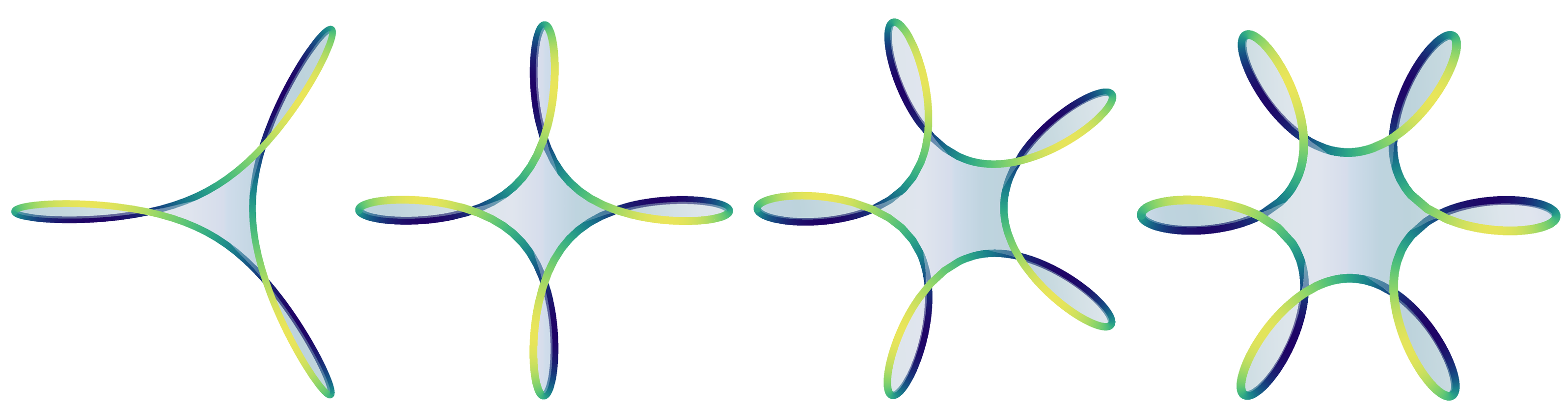

**Figure 1**. Minimal surfaces of disc type spanned by (classical) elastic curves of type $G(q,1)$. These domains are critical for $E$ with $c_o=0$ and $b=0$.

## References

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