MATH CIRCLE TTU

Number Theory

Diophantine Equations

The Problem

A store sells boxes of donuts at $7 each and pizzas at $18 each. If in one day they have received $208, how many pizzas and boxes of donuts were sold? Is that the only solution?

Diophantine Equations

A Diophantine equation $a \times x + b \times y = c$ has integer solutions (x, y) if and only if the greatest common divisor of a and b, $\gcd(a, b)$, divides c.

Example. The Diophantine equation $3x + 7y = 1$ has integer solutions.

1. What is the greatest common divisor of 3 and 7?
2. Does $\gcd(3, 7)$ divide 1?
3. Can you find a solution?

Finding All the Solutions

Consider the Diophantine equation $3x + 7y = 1$. This is the same as $3x \equiv 1 \pmod 7$. To solve this we need to "divide" by 3. How?

(i) We apply the extended Euclidean algorithm to 3 and 7. Remember?

  (a) We divide the largest number 7 by the other one 3 and obtain 2 and remainder 1, that is,
  $$7 = 2 \times 3 + 1\,.$$
  (b) Now, we divide 3 by 1 and since the remainder is 0, we are done. The greatest common divisor is the last nonzero remainder, that is, $\gcd(7, 3) = 1$.
  (c) We go backwards and get
  $$1 = 7 + (-2) \times 3\,.$$

(ii) If we take modulo in the above equation
$$1 \equiv 7 + (-2) \times 3 \pmod 7\,.$$

(iii) But, 7 is congruent to 0 modulo 7, so
$$1 \equiv (-2) \times 3 \pmod 7\,.$$

(iv) "Dividing" by 3 modulo 7 is the same as "multiplying" by -2.

(v) From $3x \equiv 1 \pmod 7$ we deduce ("dividing" by 3, which is the same as "multiplying by -2) $x \equiv -2 \pmod 7$.

(vi) Remember, $a \equiv b \pmod c \iff a = n \times c + b$. In our case, $x = -2 + 7n$.

(vii) We now solve for y: $y = 1 - 3n$.

(viii) By substituting n we obtain all the possible solutions: $(-2, 1)$, $(5, -2)$, $(-9, 4)$,...

Solve the Problem

Problem 1. A store sells boxes of donuts at \$7 each and pizzas at \$18 each. If in one day they have received \$208, how many pizzas and boxes of donuts were sold? Find all the solutions.

Another Problem

Problem 2. For a party, you are buying pizzas. You have $68 to spend and you can buy pepperoni pizzas at $12 each or cheese pizzas at $8 each. If you spent all the money, how many pizzas of each kind did you buy? Find all the possible combinations.

Ancient Chinese Problem

On a pirate ship there are 17 pirates who just stole a chest of gold coins. They try to divide these coins equally among the 17 pirates, but there are 3 left over.

The pirates begin a fight and one of them dies.

Once this pirate has died, the others calm down and try to divide all the gold coins equally again. Unfortunately, now there are 10 left over coins.

Another fight begins and another pirate dies.

After this new death, the pirates that are still alive try to divide the coins equally yet again. This time it is possible and there are no left over coins.

What is the minimum possible amount of gold coins that the pirates stole?