MATH CIRCLE TTU

Set Theory

Infinity

Hilbert's Grand Hotel

Hilbert wanted to construct the largest hotel ever, and such that it will be the largest in the future too. How many rooms should it have?

He decided to construct one with infinitely many rooms. That way, no matter what, as many guests as you can imagine could stay in his hotel.

One day, a guest arrived and, of course, there was an empty room for him.

The same night, 10 guests arrived and, of course, there were empty rooms for them.

Later on, another 100 guests arrived and, of course, there were empty rooms for them.

...

Suddenly, the hotel was full with infinitely many guests. One guest for each of the rooms.

Once the hotel was complete, a person arrived to stay during the night. It was a very cold and rainy night. How can the largest hotel ever not have space for that person? Well,... they give that person an empty room. How?

And, if 10 more people arrive later, how can you give each of them an empty room?

And, if infinitely many more people arrive, how can you accommodate them?

Working with Infinity

In order to walk from a point to another, you first need to walk half the distance, so that 1/2 of the distance is remaining. Then, you will need to walk half of the remaining distance, so that 1/4 of the distance is remaining.

If you proceed in this way, walking half of the remaining distance in each step, would you ever reach the end?

How many steps will you need?

Series

How good are you at adding?

1. Can you add the following terms?

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = ?\,.$$

   (Hint: It is related with the previous problem. How?)

2. Once you master it, can you add the following two series?

   (i)
$$(1-1) + (1-1) + (1-1) + (1-1) + (1-1) + \cdots = ?\,.$$

   (ii)
$$1 - (1-1) - (1-1) - (1-1) - (1-1) - (1-1) - \cdots = ?\,.$$

   (Hint: If the result is a real number, they must be equal.)