# Elastica Constrained Problem in Hypersurfaces of Lorentzian Space Forms*

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## Abstract

*A curve immersed in a pseudo-Riemannian manifold is called an elastic curve if it is a critical point of the bending energy [1]. The purpose of this poster is to present a few author's recent results on geodesics of hypersurfaces in a Lorentzian space form which are critical curves for the bending energy, but for variations constrained to lie on the hypersurface: the elastica constrained problem [3], [5]. First, the classification into three different types of critical geodesics for the constrained problem will be presented, in terms of their Frenet curvatures [2]. Finally, restricting ourselves to the flat Minkowski space $\mathbb{L}^3$, surfaces which are foliated by critical geodesics of each type will be studied (and classified in two of these cases) [2]. Special emphasis will be put in the warped product metric of Hashimoto surfaces [4], which are foliated by critical geodesics of the third type [2].*

## 1. Elasticae Constrained Problem

*Elastic curves* or, simply, *elasticae* are defined as those curves which are critical for the *bending energy* functional

$$\mathcal{F}(\gamma) := \int_\gamma \left( \varepsilon_2 \left\langle \frac{D\dot\gamma}{ds}, \frac{D\dot\gamma}{ds} \right\rangle + \lambda \right) ds, \tag{1}$$

where $\varepsilon_2$ is the causal character of $\frac{D\dot\gamma}{ds}$.

Now, let $\phi : M_r^{n-1} \to M_1^n(c)$ be a semi-Riemannian hypersurface of index $r$ isometrically immersed in a Lorentzian space form $M_1^n(c)$. We are interested in those curves $\gamma$ of the hypersurface which are critical points of the bending energy (1) for variations contained in $M_r^{n-1}$, the *elastica constrained problem in hypersurfaces*.

Choose two arbitrary points $p_i \in M_r^{n-1}$ and vectors $v_i \in T_{p_i}M_r^{n-1}$, $i \in \{0,1\}$, and consider the space of curves

$$\Omega = \left\{ \beta : I \to M_r^{n-1} \; s.t. \; < \frac{d^i\beta}{dt}, \frac{d^i\beta}{dt} > (t) \neq 0, \beta(i) = p_i, \frac{d\beta}{dt}(i) = v_i, i \in \{\,0,1\} \right\}, \tag{2}$$

where, $\frac{d\beta}{dt}(t)$ denotes the derivative with respect to the parameter $t \in I$, $I$ being any real interval. We wish now to analyze the variational problem associated to the energy (1) acting on $\Omega$.

From the first variation formula of $\mathcal{F}$ along $\gamma$, and due to the initial and boundary conditions of the variation we obtain the *Euler-Lagrange operator*

$$\mathcal{E}(\gamma) = 2\varepsilon_2 \frac{D^3T}{ds^3} + 3\varepsilon_1 \frac{D(\kappa^2 T)}{ds} + \varepsilon_1(2c\varepsilon_2 - \lambda)\frac{DT}{ds}. \tag{3}$$

Now, since $\gamma \subset M_r^{n-1}$ and we are taking variations in $M_r^{n-1}$, the variation field $W$ is tangent to $M_r^{n-1}$ along $\gamma$. So only the tangential part of $\mathcal{E}$ affects the first variation formula and $\gamma$ is a critical point of $\mathcal{F}$, if and only if,

$$\tan\left(\mathcal{E}(\gamma)\right) = 0\,, \tag{4}$$

where $\tan()$ denotes tangential projection on $M_r^{n-1}$.

## 2. Critical Geodesics

A *geodesic* is a constant speed curve whose tangent vector is parallel propagated along itself, i.e. a curve whose tangent, $\gamma'(t) = T(t)$, satisfies the equation $\frac{DT(t)}{dt} = 0$. Geodesics will be called *Frenet curves of rank* 1 where an immersed curve in a Lorentzian manifold $\gamma : I \to M_1^n$ is called a *Frenet curve of rank* $m$, $2 \leq m \leq n$, if $m$ is the highest integer for which there exists an orthonormal frame defined along $\gamma$, $\{e_1(t) = \gamma'(t), e_2(t), \ldots, e_m(t)\}$ and non-negative smooth functions on $\gamma$, $\kappa_i(t)$, $t \in I$, $1 \leq i \leq m-1$ (*Frenet curvatures*), such that the *Frenet-Serret equations* are satisfied. Obviously, geodesics have zero curvature.

**Proposition 1** *[2] Let $\gamma : I \to M_r^{n-1} \subset M_1^n(c)$ be a Frenet curve of rank $m$ which is geodesic of $M_r^{n-1}$. Assume that $\mathcal{F}$ is acting on $\Omega$. Then $\gamma$ is a critical point of $\mathcal{F}$ (i.e., for the hypersurface constrained problem), if and only if, one of the following cases occurs:*

*1. Rank of $\gamma$ is $1$, i.e. $\gamma$ is a geodesic of $M_1^n(c)$;*

*2. Rank of $\gamma$ is $2$, that is, the torsion of $\gamma$ vanishes, $\kappa_2 = 0$;*

*3. $\gamma$ is a Frenet curve of rank $3$ satisfying*

$$\kappa_1^2 \kappa_2 = d\,, \tag{5}$$

*where, $d \in \mathbb{R}$ is a constant and $\kappa_1$, $\kappa_2$ are the two first Frenet curvatures of $\gamma$ in $M_1^n(c)$. Moreover, in all above cases $\gamma$ lies fully in a totally geodesic submanifold $E^l \subset M_1^n(c)$ of dimension $l$= rank $\gamma$, $1 \leq m \leq 3$.*

## 3. Surfaces of $\mathbb{L}^3$ Foliated by Critical Geodesics

Consider the *Minkowsky $3$-space $\mathbb{L}^3$*, that is, the flat Lorentzian $3$-space $\mathbb{R}^3$ equipped with the metric

$$g_o = -dx_1^2 + dx_2^2 + dx_3^2\,, \tag{6}$$

where $(x_1, x_2, x_3)$ is the standard rectangular coordinate system. Now, we can study the surfaces of $\mathbb{L}^3$ foliated by critical geodesics of the three different types of Proposition 1.

- Type 1. A *ruled surface* $S$ in $3$-space $\mathbb{L}^3$ is defined by the property that it admits a parametrization $x(s,t) = \alpha(s) + tX(s)$ where $\alpha(s)$ is a connected piece of a regular curve and $X(s)$ is a nowhere vanishing vector field along the curve. Thus, rulings ($s = constant$) of $S$ are geodesics of $\mathbb{L}^3$ and ruled surfaces are examples of surfaces foliated by curves of the first type of Proposition 1.
- Type 2. A non-null unit speed curve of $\mathbb{L}^3$ with $\tau = 0$ lies in an affine plane. A curve with $\tau = 0$ is going to be called a *planar* curve. Then, we have the following result

  **Proposition 2** *[2] Let $\delta : I_1 \to \mathbb{L}^3$ be a non-null arc-length parametrized curve $\delta(t)$ in $\mathbb{L}^3$, and let $\{T_\delta(t), N_\delta(t), B_\delta(t)\}$ denote its Frenet frame. We also denote by $P_{t_o} := span\{N_\delta(t_o), B_\delta(t_o)\}$ the normal plane to $\delta(t)$ at $t_o \in I_1$.*

  *A) Suppose first that $\delta(t)$ is spacelike and take any non-null arc-length parametrized curve $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ in the timelike plane $P_{t_o}$, $\gamma : I_2 \to P_{t_o}$. Then*

  *1) If $\delta''(t)$ is non-null, define the surface $x : U = I_1 \times I_2 \to \mathbb{L}^3$ given by*

$$\begin{aligned} x(s,t) = &\ \delta(t) + \gamma_1(s)\left(\cosh\varsigma(t) N_\delta(t) + \sinh\varsigma(t) B_\delta(t)\right) \\ &+ \gamma_2(s)\left(\cosh\varsigma(t) B_\delta(t) + \sinh\varsigma(t) N_\delta(t)\right)\,, \end{aligned} \tag{7}$$

  *where $\varsigma$ satisfies $\epsilon\,\varsigma'(t) = \tau_\delta(t)$, $\tau_\delta(t)$ denotes the torsion of $\delta(t)$ and $\epsilon$ is the causal character of $\delta''(t)$. Then, the immersion $(U, x_{\mathfrak{a}})$ given in (7) defines a surface of $\mathbb{L}^3$ foliated by planar geodesics.*

  *2) If $\delta''(t)$ is null, consider the surface $x : U = I_1 \times I_2 \to \mathbb{L}^3$ defined by*

$$\begin{aligned} x_{\mathfrak{a}}(s,t) = &\ \mathfrak{a}\,\delta(t) \\ &+ \gamma_1(s)\left(\frac{1}{2d}\left(\cosh\varsigma(t) - \sinh\varsigma(t)\right) N_\delta(t) + d\left(\cosh\varsigma(t) + \sinh\varsigma(t)\right) B_\delta(t)\right) \\ &+ \gamma_2(s)\left(\frac{1}{2d}\left(\sinh\varsigma(t) - \cosh\varsigma(t)\right) N_\delta(t) + d\left(\cosh\varsigma(t) + \sinh\varsigma(t)\right) B_\delta(t)\right)\,, \end{aligned} \tag{8}$$

  *where $d \in \mathbb{R}$, $\mathfrak{a} \in \{0,1\}$ and $\varsigma'(t) = \tau_\delta(t)$ is the torsion of $\delta(t)$. Then, the immersion $(U, x_{\mathfrak{a}})$ given in (8) defines a surface of $\mathbb{L}^3$ foliated by planar geodesics of $(U, x_{\mathfrak{a}})$.*

  *Moreover, in both cases, the pseudoriemannian character of the surface is determined by that of $\gamma$, that is, $(U, x_{\mathfrak{a}})$ is Riemannian (respectively, Lorentzian) if and only if $\gamma$ is spacelike (respectively, timelike).*

  *B) Assume now that $\delta(t)$ is timelike. Take any non-null arclength parametrized curve $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ in the spacelike plane $P_{t_o}$. Then*

$$\begin{aligned} x_{\mathfrak{a}}(s,t) = &\ \mathfrak{a}\,\delta(t) + \gamma_1(s)\left(\cos\varsigma(t) N_\delta(t) - \sin\varsigma(t) B_\delta(t)\right) \\ &+ \gamma_2(s)\left(\cos\varsigma(t) B_\delta(t) + \sin\varsigma(t) N_\delta(t)\right)\,, \end{aligned} \tag{9}$$

  *where $\mathfrak{a} \in \{0,1\}$ and $\varsigma'(t) = \tau_\delta(t)$ is the torsion of $\delta(t)$. Then, the immersion $(U, x_{\mathfrak{a}})$ given in (9) defines a Lorentzian surface of $\mathbb{L}^3$ foliated by planar geodesics of $(U, x_{\mathfrak{a}})$.*

  *Conversely, locally, any surface $M_r^2$ in $\mathbb{L}^3$ foliated by non-null planar geodesics is either a ruled surface or it can be constructed as described in (7), (8) and (9).*

- Type 3. Examples of surfaces foliated by curves of the third type of Proposition 1 are given by *Hashimoto Surfaces* [2], [4].

## 4. Hashimoto Surfaces

For a *Hashimoto surface $S_\gamma$* the filament evolution $x(s,t)$ under *LIE* implies that the vortex curves ($t$-curves) $x(s,t_o)$ are geodesics in $S_\gamma$ and then $x(s,t)$ gives a parametrization of $S_\gamma$ where, as a consequence of the equivalence between the *binormal flow* and the LIE, the induced metric is a *warped product metric*,

$$g = \varepsilon_1 ds^2 + \varepsilon_3 \kappa^2 dt^2, \tag{10}$$

$\kappa$ being the curvature of $\gamma$ in $\mathbb{L}^3$. Hence, one can see that the Gauss-Codazzi equations are

$$\kappa_t = -\varepsilon_2\varepsilon_3(2\kappa_s\tau + \kappa\tau_s)\,, \tag{11}$$

$$\tau_t = \varepsilon_2\left(\varepsilon_2\frac{\kappa_{ss}}{\kappa} - \varepsilon_3\tau^2 + \frac{1}{2}\varepsilon_1\kappa^2\right)_s\,, \tag{12}$$

the *Lorentzian Da Rios equations* [2].

Lorentzian Hashimoto surfaces have the following properties (which are an extension of the Riemannian version)

**Proposition 3** *[2] With the previous notation, let $S_\gamma$ be a Hashimoto surface having by initial condition a Frenet curve of rank $2$ or $3$ in $\mathbb{L}^3$, $\gamma(s)$, parametrized by proper time. Denote by $x(s,t)$ the parametrization of $S_\gamma$ determined by LIE. Then*

*1. If all vortex curves are planar then they are elastica in either a Riemannian or a Lorentzian plane. The corrresponding Hashimoto surface is described in Propostition 2 and if $\delta''$ is not null, then $S_\gamma$ is either a right cylinder on a Lorentzian circle, or a rotation surface shaped on a planar elastica $\gamma$ of either $\mathbb{R}^2$ or $\mathbb{L}^2$.*

*2. The initial vortex curve $\gamma(s)$ evolves by rigid motions under LIE, if and only if, it is an elastica in $\mathbb{L}^3$. As a consecuence, a rank $3$ elastica $\gamma(s)$ in $\mathbb{L}^3$ evolves under LIE (by rigid motions) and the different positions of the vortex curve over time give a foliation of the assiated Hashimoto surface by $S_\gamma$-constrained elastic geodesics of type $3$ in Proposition 1.*

## References

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