Geometric Variational Problems for Curves and Surfaces

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The Principle of Least Action

Any change in nature takes place using the minimum amount of required energy.

- Often attributed to P. L. Maupertuis (1744-1746).
- Already known to G. Leibniz (1705) and L. Euler (1744).

Variational Problems for Curves (Origin)

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(G. Galilei, 1638; and R. Hooke, $\sim$1670.)

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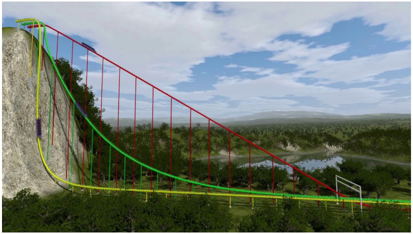

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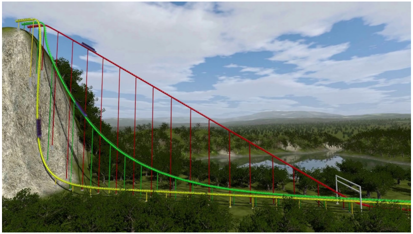

- **1697**: Johann Bernoulli, as a public challenge to Jacob Bernoulli, asked to determine the curve of minimum length (geodesics)

$$\mathcal{L}[\gamma] := \int_\gamma ds \, .$$

Variational Problems for Curves (Evolution)

- **1742**: D. Bernoulli, in a letter to L. Euler, suggested to study elastic curves as minimizers of the bending energy,

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- **1923**: W. Blaschke studied the cases $p = 1/2$ and $p = 1/3$ obtaining catenaries and parabolas, respectively, as equilibria.

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1. **(P., 2018 and 2020)**
2. **(Arroyo, Garay & P., 2018 and 2019)**
3. **(López & P., 2020)**
4. **(Palmer & P., 2020 and 2021)**

Closed Free p-Elastic Curves

Let $p \in \mathbb{R}$ and consider the functionals

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acting on the space of closed non-null smooth immersed curves in $M_r^2(\rho)$.

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The Euler-Lagrange Equation

A critical point γ of $\mathbf{\Theta}_p$ must satisfy

$$p \, \frac{d^2}{ds^2} \left(\kappa^{p-1} \right) + \epsilon_1 \epsilon_2 \, (p-1) \, \kappa^{p+1} + \epsilon_1 \, p \, \rho \, \kappa^{p-1} = 0 \,.$$

Closed Free p-Elastic Curves

Theorem

For every pair of relatively prime natural numbers (n, m) satisfying $m < 2n < \sqrt{2}m$ there exists a non-trivial closed free p-elastic curve immersed in:

- If $p > 1$, the hyperbolic plane $\mathbb{H}^2$.
- If $p \in (0, 1)$, the round sphere $\mathbb{S}^2$.
- If $p < 0$, the de Sitter 2-space $\mathbb{S}^2_1$.

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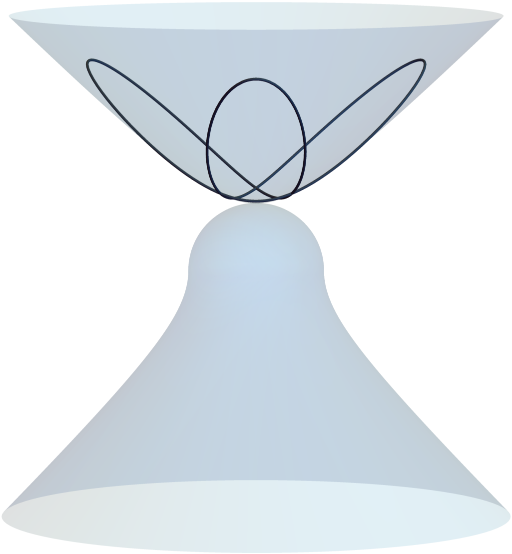
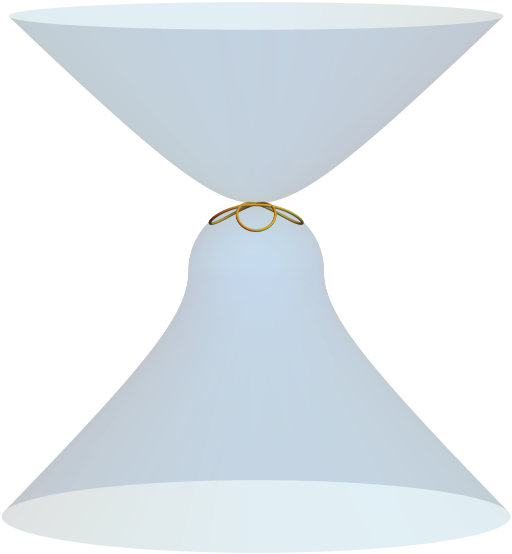
2. **(Musso & P., J. Nonlinear Sci. 2023)**
3. **(Gruber, P. & Toda, Anal. Appl. 2023)**
4. **(Montaldo & P., Commun. Anal. Geom. 2023)**
5. **(P., Samarakkody & Tran, J. Math. Anal. Appl. 2025)**

Example $p = 2$

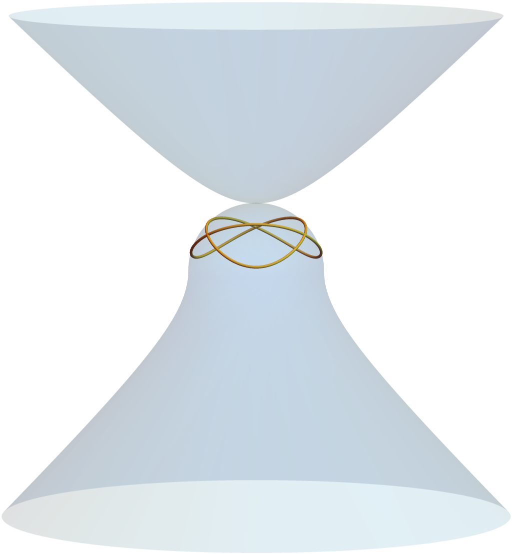


Example $p = 1.1$

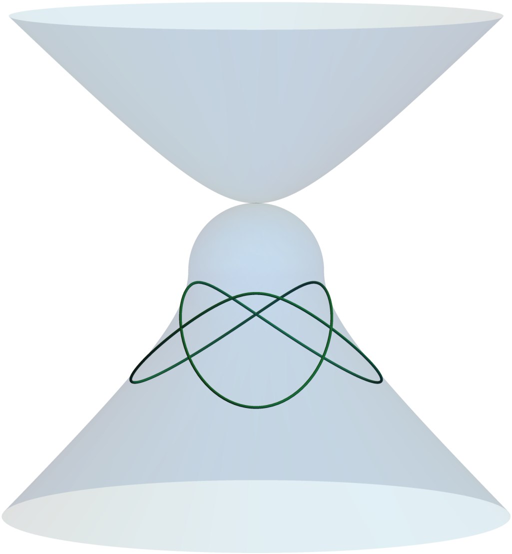
Example $p = 0.8$



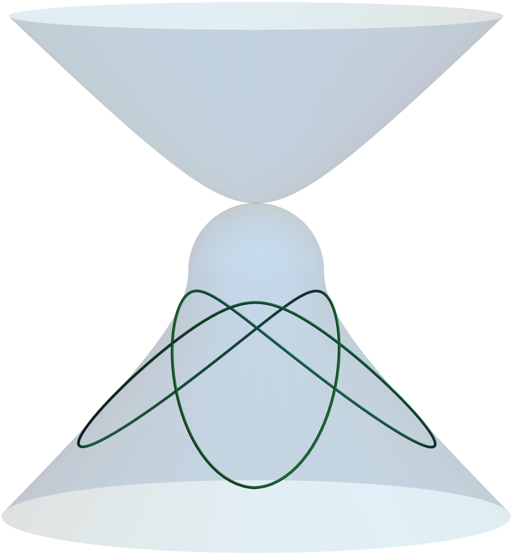
Example $p = 0.2$



Example $p = -0.5$



Example $p = -1$



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Theorem (P., Proc. Am. Math. Soc. 2024)

A planar curve γ is a translating soliton to the curvature-driven flow

$$\frac{\partial X}{\partial t}(s,t) = a\,\kappa^p(s,t)\,N(s,t)\,,$$

with $p \neq 1$ if and only if γ is p-elastic curve.

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- If $p = 1/3$ ($\alpha = 1/2$), is the suspension bridge problem.

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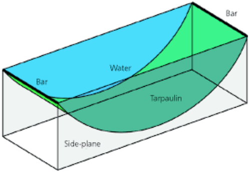

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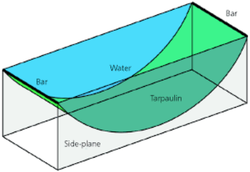

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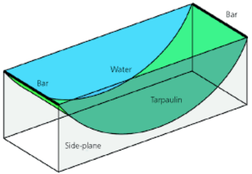

- **1744**: L. Euler described the catenoid, the minimal surface of revolution (other than the plane).
- **1760**: J. Lagrange raised the question of how to find the surface with least area

$$\mathcal{A}[\Sigma] := \int_{\Sigma} d\Sigma \,,$$

for a given fixed boundary.

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- **1811**: S. Germain proposed to study other energies such as

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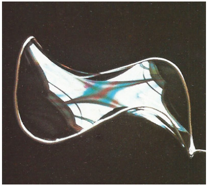

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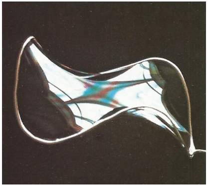

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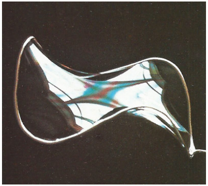

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- **1930**: J. Douglas and T. Radó found the general solution to Plateau's problem, independently.

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- **1974**: B.-Y. Chen extended the functional $\mathcal{W}$ preserving the conformal invariance.

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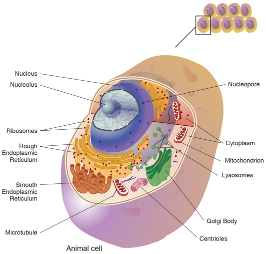
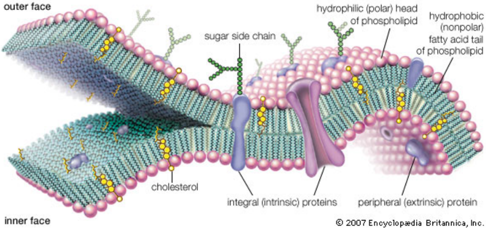
Modeling Biological Membranes





The Helfrich Energy

Let Σ be a compact (with or without boundary) surface. For an embedding $X : \Sigma \longrightarrow \mathbb{R}^3$ the Helfrich energy is given by

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The Euler-Lagrange Equation

Equilibria for $\mathcal{H}$ are characterized by

$$\Delta H + 2\,(H + c_o)\,(H[H - c_o] - K) = 0\,,$$

on the interior of Σ.

Second Order Reduction

Theorem (Palmer & P., Calc. Var. PDE 2022)

A non-CMC surface critical for $\mathcal{H}$ which contains an axially symmetric topological disc must satisfy

$$H + c_o = -\frac{\nu_3}{z}\,.$$

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Theorem (Palmer & P., Calc. Var. PDE 2022)

A sufficiently regular immersion satisfying the reduced membrane equation is critical for $\mathcal{H}$.

Second Variation Formula

Theorem (Palmer & P., J. Geom. Anal. 2024)

Let $X:\Sigma\longrightarrow\mathbb{R}^3$ be an immersion critical for $\mathcal{H}$ satisfying the reduced membrane equation. Then, for every $f\in\mathcal{C}_o^\infty(\Sigma)$ and normal variations $\delta X=f\nu$,

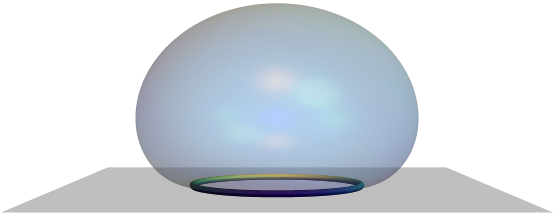
$$\delta^2\mathcal{H}[\Sigma]=\int_\Sigma f\,F[f]\,d\Sigma+\frac{1}{2}\oint_{\partial\Sigma}L[f]\partial_n f\,ds\,,$$

where

$$F[f]:=\frac{1}{2}\left(P^*+\frac{2}{z^2}\right)\circ P[f]\,.$$

(Here, P is the operator arising as twice the variation of the quantity $H+\nu_3/z$, P^* is its adjoint operator, and L comes from twice the variation of H.)

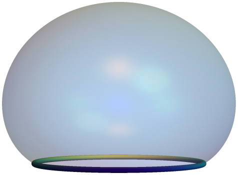
Symmetry Breaking Bifurcation



Theorem (Palmer & P., Nonlinear Anal. 2024)

Above surface Σ_0 is embedded in a one parameter family of axially symmetric solutions of the reduced membrane equation (parameterized by c_o) which all share the same boundary circle. Precisely at Σ_0, a non-axially symmetric branch bifurcates.

Axially Symmetric Family



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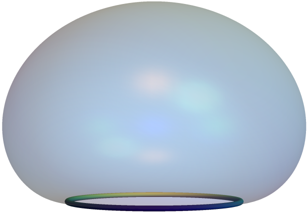

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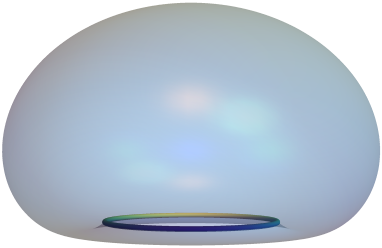
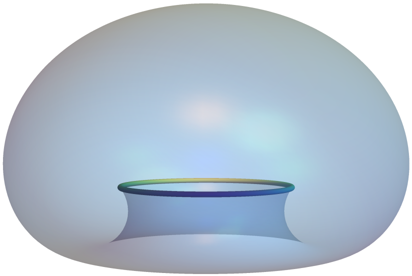
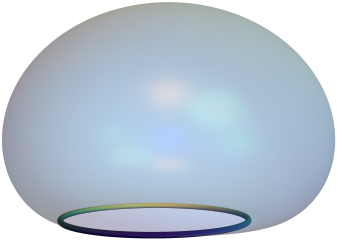


Axially Symmetric Family



Bifurcating Branch



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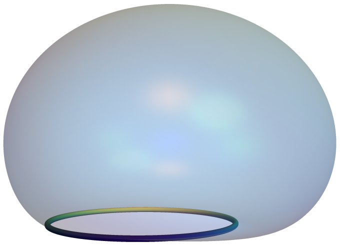

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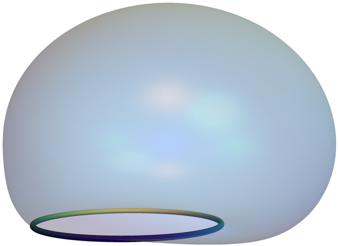

Reduced Membrane Equation

The reduced membrane equation is the Euler-Lagrange equation for

$$\mathcal{G}[\Sigma] := \int_\Sigma \frac{1}{z^2}\, d\Sigma - 2c_o \int_\Omega \frac{1}{z^2}\, dV = \widetilde{\mathcal{A}}[\Sigma] - 2c_o \int_\Omega |z|\, d\widetilde{V}\,.$$

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Applications:

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- Renormalized area in Poincaré-Einstein spaces **(P. & Tyrrell, To Be Submitted)**

Selected Publications (Since 2022)

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1. R. López and A. Pámpano, Stationary Soap Films with Vertical Potentials, Nonlinear Analysis 215 (2022), 112661.
2. B. Palmer and A. Pámpano, The Euler-Helfrich Functional, Calculus of Variations and Partial Differential Equations 61 (2022), 79.
3. S. Montaldo, C. Oniciuc and A. Pámpano, Closed Biconservative Hypersurfaces in Spheres, Journal of Mathematical Analysis and Applications 518-1 (2023), 126697.
4. E. Musso and A. Pámpano, Closed 1/2-Elasticae in the 2-Sphere, Journal of Nonlinear Science 33 (2023), 3.
5. R. López and A. Pámpano, A Relation Between Cylindrical Critical Points of Willmore-Type Energies, Weighted Areas and Vertical Potential Energies, Journal of Geometry and Physics 185 (2023), 104731.

Selected Publications (Since 2022)

6. A. Gruber, A. Pámpano and M. Toda, On p-Willmore Disks with Boundary Energies, Differential Geometry and its Applications 86 (2023), 101971.
7. E. Musso and A. Pámpano, Closed 1/2-Elasticae in the Hyperbolic Plane, Journal of Mathematical Analysis and Applications 527-1 (2023), 127388.
8. A. Gruber, A. Pámpano and M. Toda, Instability of Closed p-Elastic Curves in $\mathbb{S}^2$, Analysis and Applications 21-6 (2023), 1533-1559.
9. S. Montaldo and A. Pámpano, On the Existence of Closed Biconservative Surfaces in Space Forms, Communications in Analysis and Geometry 31-2 (2023), 291-319.
10. B. Palmer and A. Pámpano, Symmetry Breaking Bifurcation of Membranes with Boundary, Nonlinear Analysis 238 (2024), 113393.

Selected Publications (Since 2022)

11. A. Pámpano, Generalized Elastic Translating Solitons, Proceedings of the American Mathematical Society 152-4 (2024), 1743-1753.
12. B. Palmer and A. Pámpano, Stability of Membranes, Journal of Geometric Analysis 34 (2024), 328.
13. E. Musso and A. Pámpano, Integrable Flows on Null Curves in the Anti-de Sitter 3-Space, Nonlinearity 37 (2024), 115015.
14. A. Pámpano, M. Samarakkody and H. Tran, Closed p-Elastic Curves in Spheres of $\mathbb{L}^3$, Journal of Mathematical Analysis and Applications 545-2 (2025), 129147.
15. E. Musso and A. Pámpano, Geometric Transformations on Null Curves in the Anti-de Sitter 3-Space, Symmetry, Integrability and Geometry: Methods and Applications (SIGMA) 21-9 (2025), 18.

Selected Publications (Since 2022)

16. R. López, B. Palmer and A. Pámpano, Axially Symmetric Helfrich Spheres, submitted.
17. B. Palmer and A. Pámpano, Hyperbolic Geometry and the Helfrich Functional, submitted.
18. A. Pámpano and A. Tyrrell, Renormalized Area of Hypersurfaces in the Hyperbolic Space, to be submitted.
19. S. Fields, A. Pámpano and M. Samarakkody, Grim Reaper Curves in 2-Space Forms, to be submitted.

THE END

Thank You!