Geometric Variational Problems for Curves and Surfaces

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- Often attributed to P. L. Maupertuis (1744-1746).
- Already known to G. Leibniz (1705) and L. Euler (1744).

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(I) Already posed by Jordanus de Nemore (Jordan of the Forest) in the XIIIth Century.
(II) Also appears in a fundamental problem by G. Galilei (1638).
(III) History can be found in a report by R. Levien (2008).

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- L. Euler (1744): Described the shape of planar elasticae (partially solved by Jacob Bernoulli, 1692-1694).

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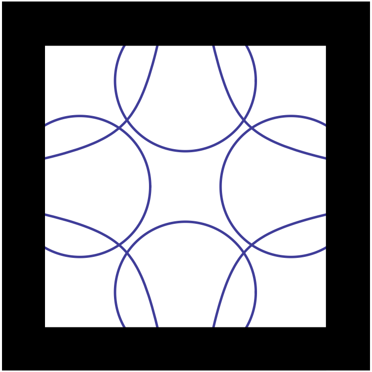
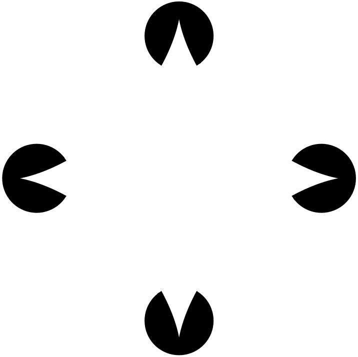
- Applications:
  (I) Image Reconstruction
  (II) Dynamics of a Vortex Filament

Image Reconstruction





(Arroyo, Garay & A. P., 2016)

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Primary Visual Cortex $V1$ (Petitot, 2003)

The unit tangent bundle $\mathbb{R}^2 \times \mathbb{S}^1$ with a suitable sub-Riemannian geometry can be used as an abstraction to study the organization and mechanisms of $V1$.

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- Geodesics are obtained by lifting minimizers in $\mathbb{R}^2$ of

$$\mathcal{F}[\gamma] := \int_\gamma \sqrt{a^2 + \kappa^2(s)}\, ds\,.$$

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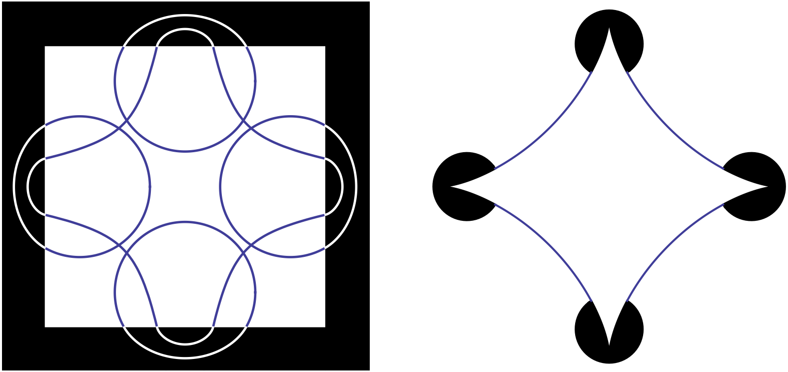

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- H. Hasimoto (1972): The LIE is equivalent to the nonlinear Schrödinger equation.

Dynamics of a Vortex Filament

- More general binormal flow for curves in $M_r^3(\rho)$, **(Garay & A. P., 2016), (Arroyo, Garay & A. P., 2017)**,

$$X_t = \dot{P}(\kappa) B\,.$$

- Using the Hasimoto transformation, equivalence with the Hirota equation. **(Garay & A. P., 2016)**

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Traveling wave solutions of the Gauss-Codazzi equations correspond with the evolution under isometries and slippage of a general Kirchhoff centerline. In particular, if there is no slippage then the initial filament is critical for

$$\mathcal{F}[\gamma] := \int_\gamma P(\kappa) \, ds \, .$$

Binormal Evolution Surfaces

(Proposal for a PhD Thesis.)

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   (Arroyo, Garay & A. P., 2018)

$$\mathcal{F}[\gamma] := \int_\gamma \sqrt{\kappa - \mu}\, ds\,.$$

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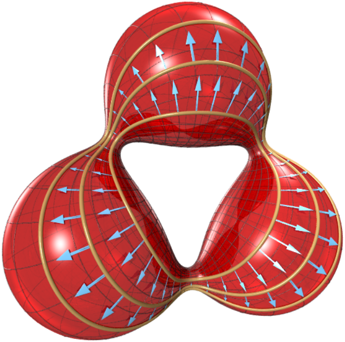
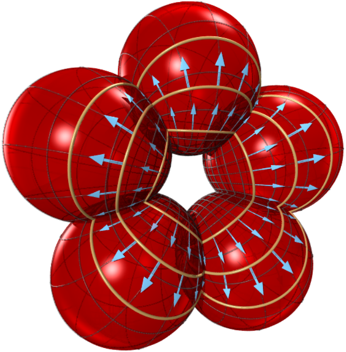
- Bour's families (isometric deformations).
- Lawson's correspondence.

Rotational CMC Surfaces in $\mathbb{S}^3(\rho)$





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Theorem **(Arroyo, Garay & A. P., 2019)**

There exist non-trivial closed critical curves in $\mathbb{S}^2(\rho)$, for any value of μ. Moreover, if the curve is also embedded, then $\mu \neq -\sqrt{\rho/3}$ is negative.

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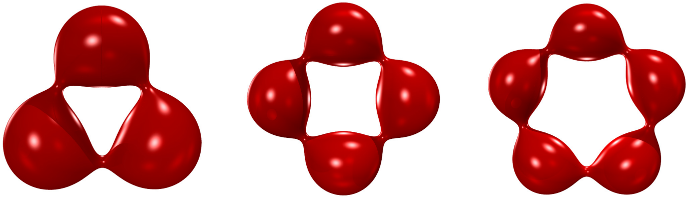

(Arroyo, Garay & A. P., 2019)

- Coincides with previous results of O. Perdomo and J.B. Ripoll.
- Verify the Lawson's conjecture (proved by S. Brendle in 2013).

Binormal Evolution Surfaces

2. Invariant Linear Weingarten Surfaces ($aH + bK = c$) in $M_r^3(\rho)$: **(A. P., 2020)**

$$\mathcal{F}[\gamma] := \int_\gamma \sqrt{\epsilon\left([\kappa - \alpha]^2 + \beta\right)}\, ds\,.$$

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3. Rotational Surfaces of Constant Astigmatism in $M^3(\rho)$: **(López & A. P., 2020)**

$$\mathcal{F}[\gamma] := \int_\gamma \kappa\, e^{\mu/\kappa}\, ds \,.$$

   - R. von Lilienthal (1887): Described these surfaces in $\mathbb{R}^3$.
   - L. Bianchi and A. Ribaucour (1872-1902): Focal surfaces have constant negative Gaussian curvature. (Collaboration.)

Binormal Evolution Surfaces

4. Rotational Linear Weingarten Surfaces ($\kappa_1 = a\kappa_2 + b$, $a \neq 1$) in $M^3(\rho)$: **(López & A. P., 2020), (A. P., 2018)**

$$\mathcal{F}[\gamma] := \int_\gamma (\kappa - \mu)^n \, ds \,.$$

   - In particular, $\mu = 0$ and $n = 1/4$ corresponds with proper biconservative surfaces. **(Montaldo & A. P., to appear)**

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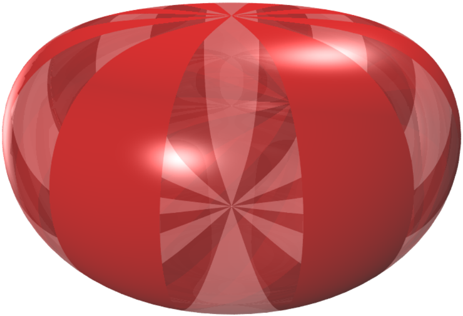
5. Rotational Constant Skew Curvature Surfaces ($\kappa_1 = \kappa_2 + c$) in $M^3(\rho)$: **(López & A. P., 2020)**

$$\mathcal{F}[\gamma] := \int_\gamma e^{\mu\kappa} \, ds \,.$$

   - They satisfy $H^2 - K = c_o^2$. In particular, circular biconcave discoids.

Circular Biconcave Discoids



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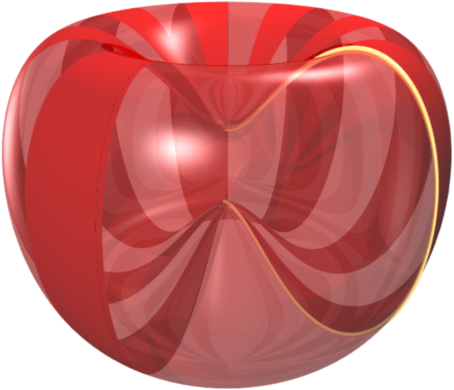

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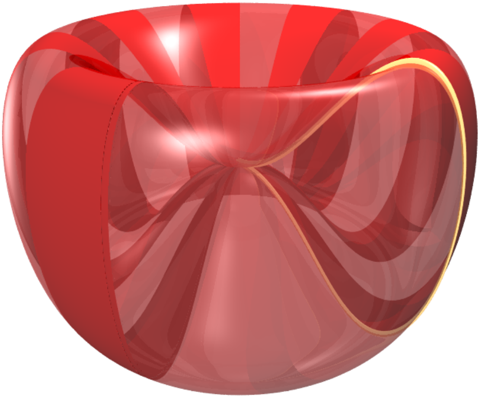

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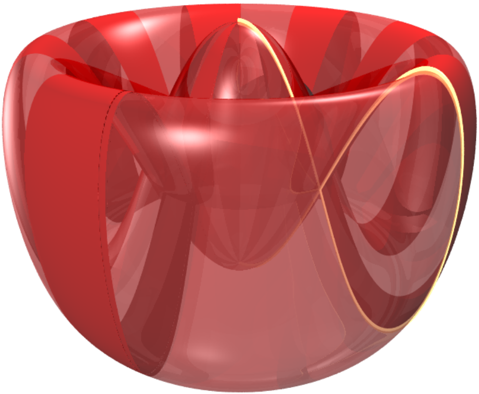

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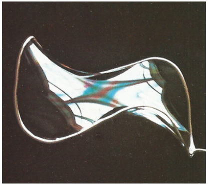

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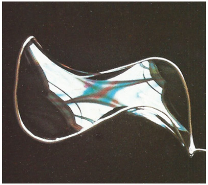

- J. Douglas and T. Radó (1930-1931): Found the general solution to Plateau's problem, independently.

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- U. Pinkall (1985): Hopf tori in $\mathbb{S}^3(\rho)$.
- F. C. Marques and A. Neves (2012): Proved the Willmore conjecture.

Modeling Biological Membranes

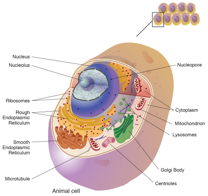
- P. B. Canham (1970): Proposed the minimization of the Willmore energy as a possible explanation for the biconcave shape of red blood cells.

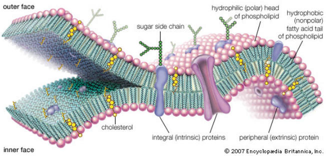
Modeling Biological Membranes

- W. Helfrich (1973): Based on liquid cristallography, suggested the extension

$$\mathcal{H}[\Sigma] := \int_{\Sigma} \left(a\,[H + c_o]^2 + bK \right) d\Sigma \,,$$

to model biological membranes.





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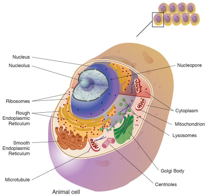

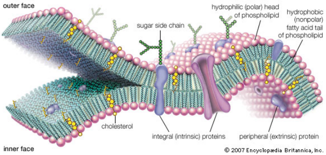

- The Euler-Lagrange equation associated to $\mathcal{H}$ is

$$\Delta H + 2\,(H + c_o)\,(H\,[H - c_o] - K) = 0\,.$$

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- They are an extension of singular minimal surfaces.

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    - (I) The Euler-Plateau Problem. **(Gruber, A. P. & Toda, 2021)**
    - (II) The Kirchhoff-Plateau Problem. **(Palmer & A. P., 2020)**
    - (III) The Euler-Helfrich Problem. **(Palmer & A. P., 2021), (Palmer & A. P., submitted)**

The Euler-Helfrich Problem

The Euler-Helfrich energy is given by:

$$E[\Sigma] := \int_{\Sigma} \left(a\,[H + c_o]^2 + bK \right) d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \beta\right) ds\,.$$

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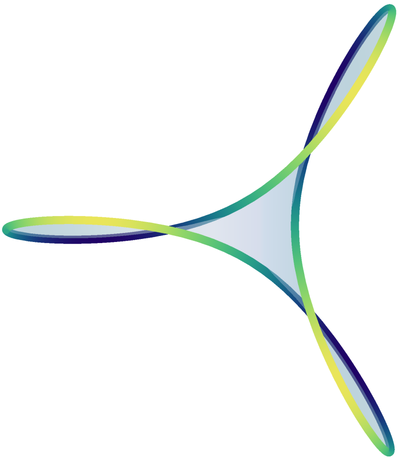
In other cases, there are no CMC critical discs.

Minimal Discs Spanned by Elastic Curves



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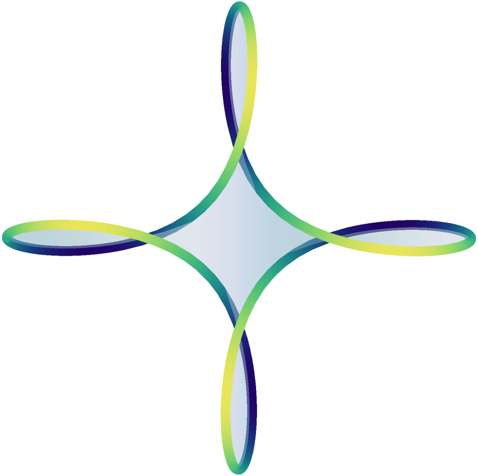

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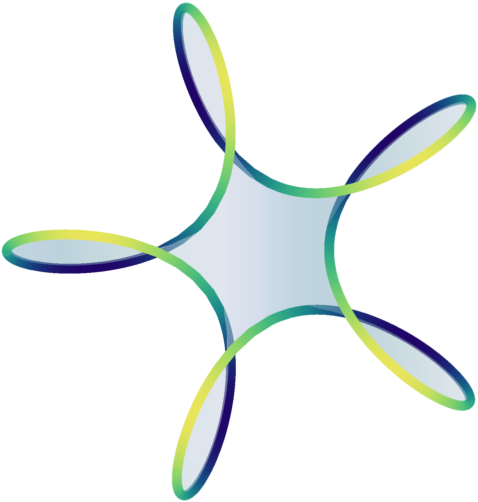

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The Euler-Helfrich Problem

Axially Symmetric (Palmer & A. P., submitted)

An axially symmetric immersion critical for E is either a part of a Delaunay surface or its mean curvature satisfies

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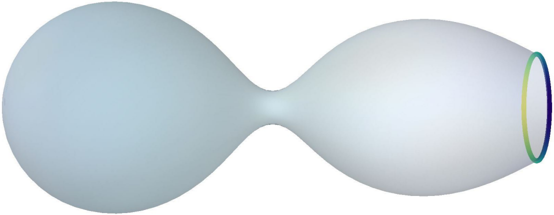

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THE END

Thank You!