Geometric Variational Problems for Curves and Surfaces

Álvaro Páampano Llarena

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Philosophical Origins

The development of Calculus was initially motivated in order to compute extrema of functions (G. Leibniz, 1684).

A natural generalization is to compute extrema of functionals (i.e., the Calculus of Variations).

The Principle of Least Action

Any change in nature takes place using the minimum amount of required energy.

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- Jacob (James) Bernoulli (1691): Proposed the problem of determining the shape of elastic rods.
- Already posed by Jordanus de Nemore (Jordan of the Forest) in the XIIIth Century.
- Also appears in a fundamental problem by G. Galilei (1638).
- History can be found in a report by R. Levien (2008).
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- **L. Euler** (1744): Described the shape of planar elasticae (partially solved by Jacob Bernoulli, 1692-1694).
Variational Problems for Curves (Recent)

- J. Langer and D. A. Singer (1984): Classified closed elastic curves in $M^2(\rho)$ and in $\mathbb{R}^3$ (torus knots).


- Multiple generalizations. For instance,$\mathcal{F}[\gamma] := \int_\gamma P(\kappa) \, ds$, for curves immersed in $M^2(\rho)$.

- Applications:
  1. Image Reconstruction
  2. Dynamics of a Vortex Filament
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Image Reconstruction

(Arroyo, Garay & A. P., 2016)
Image Reconstruction

**Primary Visual Cortex V1** (Petitot, 2003)

The unit tangent bundle $\mathbb{R}^2 \times S^1$ with a suitable sub-Riemannian geometry can be used as an abstraction to study the organization and mechanisms of V1.
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If a piece of the contour of a picture is missing, then the brain tends to complete the curve by minimizing some kind of energy, the length being the simplest one.
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- Geodesics are obtained by lifting minimizers in $\mathbb{R}^2$ of

\[
\mathcal{F}[\gamma] := \int_\gamma \sqrt{a^2 + \kappa^2(s)} \, ds .
\]
Image Reconstruction

(Arroyo, Garay & A. P., 2016)
L. S. Da Rios (1906): Modeled the movement of a vortex filament according to the localized induction equation (LIE): \[ X_t = X_s \times X_{ss} = \kappa B \].

The compatibility equations are the Gauss-Codazzi equations of the local surface generated by the evolution.

This evolution represents a binormal flow.

H. Hasimoto (1971): Found a relation between this evolution and elastic curves.

H. Hasimoto (1972): The LIE is equivalent to the nonlinear Schrödinger equation.
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Dynamics of a Vortex Filament

• More general **binormal flow** for curves in $M_r^3(\rho)$, (Garay & A. P., 2016), (Arroyo, Garay & A. P., 2017),

\[ X_t = \dot{P}(\kappa)B. \]

• Using the **Hasimoto transformation**, equivalence with the **Hirota equation**. (Garay & A. P., 2016)
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**Traveling Wave Solutions** (Garay & A. P., 2016)

Traveling wave solutions of the Gauss-Codazzi equations correspond with the evolution under **isometries and slippage** of a general Kirchhoff centerline.
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Traveling Wave Solutions (Garay & A. P., 2016)

Traveling wave solutions of the Gauss-Codazzi equations correspond with the evolution under isometries and slippage of a general Kirchhoff centerline. In particular, if there is no slippage then the initial filament is critical for

\[ \mathcal{F}[\gamma] := \int_\gamma P(\kappa) \, ds. \]
(Proposal for a PhD Thesis.)

• W. Blaschke (1921): Studied the case $\mu = 0$ in $\mathbb{R}^3$.

Delaunay Curves (Arroyo, Garay & A. P., 2018)

• Bour's families (isometric deformations).
• Lawson's correspondence.
Binormal Evolution Surfaces

(Proposal for a PhD Thesis.)

1. Invariant Constant Mean Curvature (CMC) Surfaces in $M^3(\rho)$:
   (Arroyo, Garay & A. P., 2018)

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   Critical curves are roulettes of conic foci. (Proposal for undergraduate students.)

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Rotational CMC Surfaces in $\mathbb{S}^3(\rho)$

(Arroyo, Garay & A. P., 2019)
Theorem (Arroyo, Garay & A. P., 2019)

There exist non-trivial closed critical curves in $S^3(\rho)$, for any value of $\mu$. Moreover, if the curve is also embedded, then $\mu \neq -\sqrt{\rho/3}$ is negative.

• Coincides with previous results of O. Perdomo and J.B. Ripoll.

• Verify the Lawson’s conjecture (proved by S. Brendle in 2013).
CMC Tori in $S^3(\rho)$

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- Coincides with previous results of O. Perdomo and J.B. Ripoll.
- Verify the Lawson’s conjecture (proved by S. Brendle in 2013).
2. Invariant Linear Weingarten Surfaces \((aH + bK = c)\) in 
\(M^3_r(\rho)\): (A. P., 2020)

\[ F[\gamma] := \int_\gamma \sqrt{\epsilon \left( [\kappa - \alpha]^2 + \beta \right)} \, ds. \]

- In particular, rotational constant Gaussian curvature surfaces
appear for \(a = \alpha = 0\). (Image Reconstruction).
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3. Rotational Surfaces of Constant Astigmatism in \(M^3(\rho)\):

(López & A. P., 2020)

\[
\mathcal{F}[\gamma] := \int_\gamma \kappa \, e^{\mu/\kappa} \, ds.
\]

- R. von Lilienthal (1887): Described these surfaces in \(\mathbb{R}^3\).
- L. Bianchi and A. Ribaucour (1872-1902): Focal surfaces have constant negative Gaussian curvature. (Collaboration.)
Binormal Evolution Surfaces

4. Rotational Linear Weingarten Surfaces \((\kappa_1 = a\kappa_2 + b, a \neq 1)\) in \(M^3(\rho)\): (López & A. P., 2020), (A. P., 2018)

\[
\mathcal{F}[\gamma] := \int_{\gamma} (\kappa - \mu)^n \, ds.
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- In particular, \(\mu = 0\) and \(n = 1/4\) corresponds with proper biconservative surfaces. (Montaldo & A. P., to appear)
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5. Rotational Constant Skew Curvature Surfaces \( (\kappa_1 = \kappa_2 + c) \) in \( \mathcal{M}^3(\rho) \): (López & A. P., 2020)

\[
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- They satisfy \( H^2 - K = c_o^2 \). In particular, circular biconcave discoids.
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Variational Problems for Surfaces (Area)

• J. Lagrange (1760): Raised the question of how to find the surface with least area $A = \int \Sigma d\Sigma$, for a given fixed boundary.

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T. J. Willmore (1968): Reintroduced the functional \( \mathcal{W} \) and stated his famous conjecture.

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Modeling Biological Membranes

- **P. B. Canham** (1970): Proposed the minimization of the Willmore energy as a possible explanation for the biconcave shape of red blood cells.
W. Helfrich (1973): Based on liquid cristallography, suggested the extension

\[ \mathcal{H}[\Sigma] := \int_{\Sigma} \left( a[H + c_0]^2 + bK \right) d\Sigma, \]

to model biological membranes.
• **W. Helfrich** (1973): Based on liquid crystallography, suggested the extension

\[ H[\Sigma] := \int_{\Sigma} \left( a[H + c_o]^2 + bK \right) d\Sigma , \]

to model biological membranes.

• The Euler-Lagrange equation associated to \( H \) is

\[ \Delta H + 2 (H + c_o) (H [H - c_o] - K) = 0 . \]
The Helfrich Energy

There are some solutions:

1. **Constant Mean Curvature Surfaces** with $H \equiv -c_o$.

   - Axially Symmetric Discs (submitted)
The Helfrich Energy

There are some solutions:

1. **Constant Mean Curvature Surfaces** with \( H \equiv -c_o \).
2. **Circular Biconcave Discoids** (far from the axis of rotation).

Axially Symmetric Discs

An axially symmetric disc critical for \( H \) must satisfy \( H + c_o = -\nu z \).

- They are an extension of singular minimal surfaces.
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Boundary Problems

Different problems depending on the nature of $\partial \Sigma$:

• The Free Boundary Problem. The boundary $\partial \Sigma$ lies in a fixed supporting surface.
• The Fixed Boundary Problem. The boundary $\partial \Sigma$ is prescribed and immovable.
• The Thread Problem. Only the length of the boundary $\partial \Sigma$ is prescribed.
• Combinations of energies. The boundary components of $\partial \Sigma$ are elastic.
  (i) The Euler-Plateau Problem. (Gruber, A. P. & Toda, 2021)
  (iii) The Euler-Helfrich Problem. (Palmer & A. P., submitted)
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The Euler-Helfrich Problem

The **Euler-Helfrich energy** is given by:

\[ E[\Sigma] := \int_{\Sigma} \left( a[H + c_o]^2 + bK \right) d\Sigma + \oint_{\partial\Sigma} (\alpha \kappa^2 + \beta) \, ds. \]
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**CMC Discs** (Palmer & A. P., 2021)

Consider a CMC disc critical for the energy $E$. Then:

1. Case $b \neq 0$ and $c_0 = 0$. The surface is a spherical cap.
The Euler-Helfrich energy is given by:

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**CMC Discs** (Palmer & A. P., 2021)

Consider a CMC disc critical for the energy \( E \). Then:

1. Case \( b \neq 0 \) and \( c_o = 0 \). The surface is a spherical cap.
2. Case \( b = 0 \). The boundary is an unknotted closed elastic curve.
The Euler-Helfrich Problem

The Euler-Helfrich energy is given by:

\[ E[\Sigma] := \int_{\Sigma} \left( a[H + c_o]^2 + bK \right) d\Sigma + \oint_{\partial \Sigma} (\alpha \kappa^2 + \beta) \, ds. \]

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In other cases, there are no CMC critical discs.
Minimal Discs Spanned by Elastic Curves

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An axially symmetric immersion critical for $E$ is either a part of a Delaunay surface or its mean curvature satisfies

$$H + c_o = -\frac{\nu_3}{z}.$$
The Euler-Helfrich Problem

**Axially Symmetric** (Palmer & A. P., submitted)

An axially symmetric immersion critical for $E$ is either a part of a Delaunay surface or its mean curvature satisfies

$$H + c_o = -\frac{\nu_3}{z}.$$
THE END

Thank You!