Minimal Surfaces Spanning a Twisted Elastic Ribbon

Álvaro Pámpano Llarena

Elasticity Seminar
Texas Tech University

Lubbock, September 16 (2020)

Minimal Surfaces

Let $X : \Sigma \to \mathbb{R}^3$ be an immersion of an oriented surface Σ.

Minimal Surfaces

Let $X : \Sigma \to \mathbb{R}^3$ be an immersion of an oriented surface Σ. The area (functional) is given by

$$\mathcal{A}[X] := \int_\Sigma d\Sigma \,.$$

Minimal Surfaces

Let $X : \Sigma \to \mathbb{R}^3$ be an immersion of an oriented surface Σ. The area (functional) is given by

$$\mathcal{A}[X] := \int_\Sigma d\Sigma \, .$$

Definition

The immersion $X : \Sigma \to \mathbb{R}^3$ is said to be minimal (or the surface Σ is said to be minimal) if it is a critical point of $\mathcal{A}[X]$.

Minimal Surfaces

Let $X : \Sigma \to \mathbb{R}^3$ be an immersion of an oriented surface Σ. The area (functional) is given by

$$\mathcal{A}[X] := \int_\Sigma d\Sigma \,.$$

Definition

The immersion $X : \Sigma \to \mathbb{R}^3$ is said to be minimal (or the surface Σ is said to be minimal) if it is a critical point of $\mathcal{A}[X]$.

For compactly supported variations δX,

$$\delta\mathcal{A}[X] = -2\int_\Sigma H\nu \cdot \delta X \, d\Sigma \,,$$

where H is the mean curvature and ν is the unit normal to Σ.

Minimal Surfaces

Let $X : \Sigma \to \mathbb{R}^3$ be an immersion of an oriented surface Σ. The area (functional) is given by

$$\mathcal{A}[X] := \int_\Sigma d\Sigma\,.$$

Definition

The immersion $X : \Sigma \to \mathbb{R}^3$ is said to be minimal (or the surface Σ is said to be minimal) if it is a critical point of $\mathcal{A}[X]$.

For compactly supported variations δX,

$$\delta\mathcal{A}[X] = -2\int_\Sigma H\nu \cdot \delta X\, d\Sigma\,,$$

where H is the mean curvature and ν is the unit normal to Σ. So, the immersion is minimal if and only if $H \equiv 0$.

Minimal Surfaces with Boundary

Assume that Σ is an oriented, compact surface with boundary $\partial\Sigma$ (positively oriented).

Minimal Surfaces with Boundary

Assume that Σ is an oriented, compact surface with boundary $\partial\Sigma$ (positively oriented).

Different problems depending on the nature of $\partial\Sigma$:

Minimal Surfaces with Boundary

Assume that Σ is an oriented, compact surface with boundary $\partial\Sigma$ (positively oriented).

Different problems depending on the nature of $\partial\Sigma$:

- The Free Boundary Problem. The boundary $\partial\Sigma$ lies in a fixed supporting surface.

Minimal Surfaces with Boundary

Assume that Σ is an oriented, compact surface with boundary $\partial\Sigma$ (positively oriented).

Different problems depending on the nature of $\partial\Sigma$:

- The Free Boundary Problem. The boundary $\partial\Sigma$ lies in a fixed supporting surface.
- The Plateau Problem. The boundary $\partial\Sigma$ is prescribed and immovable (fixed boundary).

Minimal Surfaces with Boundary

Assume that Σ is an oriented, compact surface with boundary $\partial\Sigma$ (positively oriented).

Different problems depending on the nature of $\partial\Sigma$:

- The Free Boundary Problem. The boundary $\partial\Sigma$ lies in a fixed supporting surface.
- The Plateau Problem. The boundary $\partial\Sigma$ is prescribed and immovable (fixed boundary).
- The Thread Problem. Only the length of the boundary $\partial\Sigma$ is prescribed.

Minimal Surfaces with Boundary

Assume that Σ is an oriented, compact surface with boundary $\partial\Sigma$ (positively oriented).

Different problems depending on the nature of $\partial\Sigma$:

- The Free Boundary Problem. The boundary $\partial\Sigma$ lies in a fixed supporting surface.
- The Plateau Problem. The boundary $\partial\Sigma$ is prescribed and immovable (fixed boundary).
- The Thread Problem. Only the length of the boundary $\partial\Sigma$ is prescribed.
- The Euler-Plateau Problem. The boundary components of $\partial\Sigma$ are elastic

Minimal Surfaces with Boundary

Assume that Σ is an oriented, compact surface with boundary $\partial\Sigma$ (positively oriented).

Different problems depending on the nature of $\partial\Sigma$:

- The Free Boundary Problem. The boundary $\partial\Sigma$ lies in a fixed supporting surface.
- The Plateau Problem. The boundary $\partial\Sigma$ is prescribed and immovable (fixed boundary).
- The Thread Problem. Only the length of the boundary $\partial\Sigma$ is prescribed.
- The Euler-Plateau Problem. The boundary components of $\partial\Sigma$ are elastic: **(Giomi & Mahadevan, 2012)**

$$\mathcal{EP}[X] := \sigma\mathcal{A}[X] + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \beta\right) ds\,.$$

(Extension of) the Euler-Plateau Problem

(Extension of) the Euler-Plateau Problem

The Euler-Plateau problem combines two of the oldest objects in Differential Geometry

(Extension of) the Euler-Plateau Problem

The Euler-Plateau problem combines two of the oldest objects in Differential Geometry:

- Elastic Curves,

(Extension of) the Euler-Plateau Problem

The Euler-Plateau problem combines two of the oldest objects in Differential Geometry:

- Elastic Curves, and
- Minimal Surfaces.

(Extension of) the Euler-Plateau Problem

The Euler-Plateau problem combines two of the oldest objects in Differential Geometry:

- Elastic Curves, and
- Minimal Surfaces.

However, physically, it can be improved: including the twisting of the boundary $\partial\Sigma$.

(Extension of) the Euler-Plateau Problem

The Euler-Plateau problem combines two of the oldest objects in Differential Geometry:

- Elastic Curves, and
- Minimal Surfaces.

However, physically, it can be improved: including the twisting of the boundary $\partial\Sigma$.

The Kirchhoff-Plateau Problem (Biria & Fried, 2014)

The boundary $\partial\Sigma$ is treated as a thin elastic (flexible) rod, which is allowed to twist.

(Extension of) the Euler-Plateau Problem

The Euler-Plateau problem combines two of the oldest objects in Differential Geometry:

- Elastic Curves, and
- Minimal Surfaces.

However, physically, it can be improved: including the twisting of the boundary $\partial\Sigma$.

The Kirchhoff-Plateau Problem (Biria & Fried, 2014)

The boundary $\partial\Sigma$ is treated as a thin elastic (flexible) rod, which is allowed to twist.
The twisting is measured by including another term in the energy which depends on a choice of an orthonormal framing (Kirchhoff elastic rod).

Kirchhoff Elastic Rods (Introduction)

Kirchhoff Elastic Rods (Introduction)

Physical Definition

A Kirchhoff elastic rod is a thin elastic rod with circular cross sections and uniform density, naturally straight and prismatic when unstressed and which is being held bent and twisted by external forces and moments acting at its ends alone.

Kirchhoff Elastic Rods (Introduction)

Physical Definition

A Kirchhoff elastic rod is a thin elastic rod with circular cross sections and uniform density, naturally straight and prismatic when unstressed and which is being held bent and twisted by external forces and moments acting at its ends alone.

Mathematically, the usual formulation is a variational problem which includes a curve (the center line)

Kirchhoff Elastic Rods (Introduction)

Physical Definition

A Kirchhoff elastic rod is a thin elastic rod with circular cross sections and uniform density, naturally straight and prismatic when unstressed and which is being held bent and twisted by external forces and moments acting at its ends alone.

Mathematically, the usual formulation is a variational problem which includes a curve (the center line) and an orthonormal frame (the material frame)

Kirchhoff Elastic Rods (Introduction)

Physical Definition

A Kirchhoff elastic rod is a thin elastic rod with circular cross sections and uniform density, naturally straight and prismatic when unstressed and which is being held bent and twisted by external forces and moments acting at its ends alone.

Mathematically, the usual formulation is a variational problem which includes a curve (the center line) and an orthonormal frame (the material frame):

- The Center Line. An elastic space curve (bending energy).

Kirchhoff Elastic Rods (Introduction)

Physical Definition

A Kirchhoff elastic rod is a thin elastic rod with circular cross sections and uniform density, naturally straight and prismatic when unstressed and which is being held bent and twisted by external forces and moments acting at its ends alone.

Mathematically, the usual formulation is a variational problem which includes a curve (the center line) and an orthonormal frame (the material frame):

- The Center Line. An elastic space curve (bending energy).
- The Material Frame. The square of the norm of its derivative in the normal bundle (twisting energy).

Kirchhoff Elastic Rods (Energy)

Kirchhoff Elastic Rods (Energy)

Let $C : [0, L] \to \mathbb{R}^3$ be an arc-length parameterized curve, which represents the center line,

Kirchhoff Elastic Rods (Energy)

Let $C : [0, L] \to \mathbb{R}^3$ be an arc-length parameterized curve, which represents the center line, and let $M := \{M_1, M_2\}$ denote the material frame.

Kirchhoff Elastic Rods (Energy)

Let $C : [0, L] \to \mathbb{R}^3$ be an arc-length parameterized curve, which represents the center line, and let $M := \{M_1, M_2\}$ denote the material frame.

Variational Problem (Langer & Singer, 1986)

The energy of an inextensible Kirchhoff elastic rod is given by

$$\mathcal{K}[(C, M)] := \int_C \left(\alpha\kappa^2 + \varpi\|\nabla^\perp M_1\|^2 + \beta\right) ds\,,$$

where κ denotes the (Frenet) curvature of C.

Kirchhoff Elastic Rods (Energy)

Let $C : [0, L] \to \mathbb{R}^3$ be an arc-length parameterized curve, which represents the center line, and let $M := \{M_1, M_2\}$ denote the material frame.

Variational Problem (Langer & Singer, 1986)

The energy of an inextensible Kirchhoff elastic rod is given by

$$\mathcal{K}[(C, M)] := \int_C \left(\alpha\kappa^2 + \varpi\|\nabla^\perp M_1\|^2 + \beta\right) ds\,,$$

where κ denotes the (Frenet) curvature of C.

- Obviously, $\varpi = 0$ (non-shearable rod) reduces to the classical bending energy.

The Kirchhoff-Plateau Problem

The Kirchhoff-Plateau Problem

Let $X : \Sigma \to \mathbb{R}^3$, then the Kirchhoff-Plateau energy is

$$\mathcal{KP}[(X, M)] := \sigma \mathcal{A}[X] + \mathcal{K}[(X|_{\partial\Sigma}, M)] \, .$$

The Kirchhoff-Plateau Problem

Let $X : \Sigma \to \mathbb{R}^3$, then the Kirchhoff-Plateau energy is

$$\mathcal{KP}[(X, M)] := \sigma \mathcal{A}[X] + \mathcal{K}[(X|_{\partial\Sigma}, M)] \, .$$

Or, equivalently, if we expand each term,

$$\mathcal{KP}[(X, M)] = \sigma \int_{\Sigma} d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \varpi \|\nabla^{\perp} M_1\|^2 + \beta \right) ds \, .$$

The Kirchhoff-Plateau Problem

Let $X : \Sigma \to \mathbb{R}^3$, then the Kirchhoff-Plateau energy is

$$\mathcal{KP}[(X, M)] := \sigma \mathcal{A}[X] + \mathcal{K}[(X|_{\partial\Sigma}, M)] \,.$$

Or, equivalently, if we expand each term,

$$\mathcal{KP}[(X, M)] = \sigma \int_{\Sigma} d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \varpi \|\nabla^{\perp} M_1\|^2 + \beta \right) ds \,.$$

- The energy $\mathcal{KP}$ depends on both the surface immersion X and the material frame M.

The Kirchhoff-Plateau Problem

Let $X : \Sigma \to \mathbb{R}^3$, then the Kirchhoff-Plateau energy is

$$\mathcal{KP}[(X, M)] := \sigma \mathcal{A}[X] + \mathcal{K}[(X|_{\partial\Sigma}, M)] .$$

Or, equivalently, if we expand each term,

$$\mathcal{KP}[(X, M)] = \sigma \int_{\Sigma} d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \varpi \|\nabla^{\perp} M_1\|^2 + \beta \right) ds .$$

- The energy $\mathcal{KP}$ depends on both the surface immersion X and the material frame M.
- By varying independently, the normal frame we obtain that

  $$\nabla^{\perp} M_1 \cdot M_2$$

  is constant.

The Kirchhoff-Plateau Problem

Let $X : \Sigma \to \mathbb{R}^3$, then the Kirchhoff-Plateau energy is

$$\mathcal{KP}[(X,M)] := \sigma\mathcal{A}[X] + \mathcal{K}[(X|_{\partial\Sigma}, M)]\,.$$

Or, equivalently, if we expand each term,

$$\mathcal{KP}[(X,M)] = \sigma\int_\Sigma d\Sigma + \oint_{\partial\Sigma}\left(\alpha\kappa^2 + \varpi\|\nabla^\perp M_1\|^2 + \beta\right) ds\,.$$

- The energy $\mathcal{KP}$ depends on both the surface immersion X and the material frame M.
- By varying independently, the normal frame we obtain that

  $$\nabla^\perp M_1 \cdot M_2$$

  is constant. This yields to the total (Frenet) torsion term.

Twisted Elastic Ribbon

Twisted Elastic Ribbon

We will consider the Kirchhoff-Plateau energy where the material frame (fixed from the beginning) is the normal part of the Darboux frame.

Twisted Elastic Ribbon

We will consider the Kirchhoff-Plateau energy where the material frame (fixed from the beginning) is the normal part of the Darboux frame. The Darboux frame of $\partial\Sigma$ is the orthonormal frame $\{T, \nu, n\}$, where:

- T is the tangent vector field to $\partial\Sigma$,

Twisted Elastic Ribbon

We will consider the Kirchhoff-Plateau energy where the material frame (fixed from the beginning) is the normal part of the Darboux frame. The Darboux frame of $\partial\Sigma$ is the orthonormal frame $\{T, \nu, n\}$, where:

- T is the tangent vector field to $\partial\Sigma$,
- ν is the unit normal vector field to Σ,

Twisted Elastic Ribbon

We will consider the Kirchhoff-Plateau energy where the material frame (fixed from the beginning) is the normal part of the Darboux frame. The Darboux frame of $\partial\Sigma$ is the orthonormal frame $\{T, \nu, n\}$, where:

- T is the tangent vector field to $\partial\Sigma$,
- ν is the unit normal vector field to Σ, and
- $n := T \times \nu$ is the (outward pointing) conormal vector field.

Twisted Elastic Ribbon

We will consider the Kirchhoff-Plateau energy where the material frame (fixed from the beginning) is the normal part of the Darboux frame. The Darboux frame of $\partial\Sigma$ is the orthonormal frame $\{T, \nu, n\}$, where:

- T is the tangent vector field to $\partial\Sigma$,
- ν is the unit normal vector field to Σ, and
- $n := T \times \nu$ is the (outward pointing) conormal vector field.

Note that the energy only depends on the immersion X.

Twisted Elastic Ribbon

We will consider the Kirchhoff-Plateau energy where the material frame (fixed from the beginning) is the normal part of the Darboux frame. The Darboux frame of $\partial\Sigma$ is the orthonormal frame $\{T, \nu, n\}$, where:

- T is the tangent vector field to $\partial\Sigma$,
- ν is the unit normal vector field to Σ, and
- $n := T \times \nu$ is the (outward pointing) conormal vector field.

Note that the energy only depends on the immersion X.

Motivation:

- Mathematical Motivation. The Darboux frame captures how a curve is contained in the surface.

Twisted Elastic Ribbon

We will consider the Kirchhoff-Plateau energy where the material frame (fixed from the beginning) is the normal part of the Darboux frame. The Darboux frame of $\partial\Sigma$ is the orthonormal frame $\{T, \nu, n\}$, where:

- T is the tangent vector field to $\partial\Sigma$,
- ν is the unit normal vector field to Σ, and
- $n := T \times \nu$ is the (outward pointing) conormal vector field.

Note that the energy only depends on the immersion X.

Motivation:

- Mathematical Motivation. The Darboux frame captures how a curve is contained in the surface.
- Physical Motivation. Replace the boundary rod with a boundary ribbon.

The Total Energy

The Total Energy

Let $\tau_g := n' \cdot \nu$ be the geodesic torsion of the boundary. Then, for an immersion $X : \Sigma \to \mathbb{R}^3$ our total energy is

$$E[X] := \sigma \int_{\Sigma} d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \varpi{\tau_g}^2 + \beta\right) ds \,,$$

The Total Energy

Let $\tau_g := n' \cdot \nu$ be the geodesic torsion of the boundary. Then, for an immersion $X : \Sigma \to \mathbb{R}^3$ our total energy is

$$E[X] := \sigma \int_\Sigma d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \varpi{\tau_g}^2 + \beta\right) ds \,,$$

where the energy parameters represent:

- $\sigma > 0$ is the surface tension,

The Total Energy

Let $\tau_g := n' \cdot \nu$ be the geodesic torsion of the boundary. Then, for an immersion $X : \Sigma \to \mathbb{R}^3$ our total energy is

$$E[X] := \sigma \int_\Sigma d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \varpi{\tau_g}^2 + \beta\right) ds \,,$$

where the energy parameters represent:

- $\sigma > 0$ is the surface tension,
- $\alpha > 0$ is the flexural rigidity,

The Total Energy

Let $\tau_g := n' \cdot \nu$ be the geodesic torsion of the boundary. Then, for an immersion $X : \Sigma \to \mathbb{R}^3$ our total energy is

$$E[X] := \sigma \int_\Sigma d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \varpi{\tau_g}^2 + \beta\right) ds\,,$$

where the energy parameters represent:

- $\sigma > 0$ is the surface tension,
- $\alpha > 0$ is the flexural rigidity,
- $\varpi > 0$ is the torsional rigidity,

The Total Energy

Let $\tau_g := n' \cdot \nu$ be the geodesic torsion of the boundary. Then, for an immersion $X : \Sigma \to \mathbb{R}^3$ our total energy is

$$E[X] := \sigma \int_\Sigma d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \varpi{\tau_g}^2 + \beta\right) ds \,,$$

where the energy parameters represent:

- $\sigma > 0$ is the surface tension,
- $\alpha > 0$ is the flexural rigidity,
- $\varpi > 0$ is the torsional rigidity, and
- $\beta \in \mathbb{R}$ is the edge tension.

The Total Energy

Let $\tau_g := n' \cdot \nu$ be the geodesic torsion of the boundary. Then, for an immersion $X : \Sigma \to \mathbb{R}^3$ our total energy is

$$E[X] := \sigma \int_\Sigma d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \varpi{\tau_g}^2 + \beta\right) ds \,,$$

where the energy parameters represent:

- $\sigma > 0$ is the surface tension,
- $\alpha > 0$ is the flexural rigidity,
- $\varpi > 0$ is the torsional rigidity, and
- $\beta \in \mathbb{R}$ is the edge tension.

Observe that α and ϖ are related by the Poisson's ratio ϵ ($\epsilon \in [-1, 1/2]$),

$$\alpha = (1+\epsilon)\,\varpi \,.$$

Euler-Lagrange Equations (1)

Euler-Lagrange Equations (1)

1. We first consider compactly supported variations, obtaining

$$H \equiv 0 \,, \qquad \text{on } \Sigma \,,$$

Euler-Lagrange Equations (1)

1. We first consider compactly supported variations, obtaining

$$H \equiv 0 \,, \qquad\qquad \text{on } \Sigma \,,$$

i.e. a critical immersion must be minimal.

Euler-Lagrange Equations (1)

1. We first consider compactly supported variations, obtaining

$$H \equiv 0\,, \qquad \text{on } \Sigma\,,$$

   i.e. a critical immersion must be minimal.

2. We take now normal variations $\delta X = \psi\nu$

Euler-Lagrange Equations (1)

1. We first consider compactly supported variations, obtaining

$$H \equiv 0 \,, \qquad \text{on } \Sigma \,,$$

   i.e. a critical immersion must be minimal.

2. We take now normal variations $\delta X = \psi\nu$ to get along the boundary $\partial\Sigma$,

$$0 = \oint_{\partial\Sigma} \left(J' \cdot [\psi\nu] - 2\varpi\tau_g' \partial_n \psi \right) ds \,,$$

   where $J := 2\alpha T'' + \left(3\alpha\kappa^2 + \varpi\tau_g^2 - \beta\right) T + 2\varpi\tau_g T \times T'$.

Euler-Lagrange Equations (1)

1. We first consider compactly supported variations, obtaining

$$H \equiv 0\,, \qquad \text{on } \Sigma\,,$$

   i.e. a critical immersion must be minimal.

2. We take now normal variations $\delta X = \psi\nu$ to get along the boundary $\partial\Sigma$,

$$0 = \oint_{\partial\Sigma} \left(J' \cdot [\psi\nu] - 2\varpi\tau_g'\partial_n\psi\right) ds\,,$$

   where $J := 2\alpha T'' + \left(3\alpha\kappa^2 + \varpi\tau_g^2 - \beta\right) T + 2\varpi\tau_g T \times T'$.
   This implies:

   - $J' \cdot \nu \equiv 0$ on $\partial\Sigma$,

Euler-Lagrange Equations (1)

1. We first consider compactly supported variations, obtaining

$$H \equiv 0\,, \qquad \text{on } \Sigma\,,$$

   i.e. a critical immersion must be minimal.

2. We take now normal variations $\delta X = \psi\nu$ to get along the boundary $\partial\Sigma$,

$$0 = \oint_{\partial\Sigma} \left(J' \cdot [\psi\nu] - 2\varpi\tau_g' \partial_n\psi\right) ds\,,$$

   where $J := 2\alpha T'' + \left(3\alpha\kappa^2 + \varpi\tau_g^2 - \beta\right) T + 2\varpi\tau_g T \times T'$.
   This implies:

   - $J' \cdot \nu \equiv 0$ on $\partial\Sigma$, and
   - τ_g is (locally) constant.

Euler-Lagrange Equations (1)

1. We first consider compactly supported variations, obtaining

$$H \equiv 0\,, \qquad \text{on } \Sigma\,,$$

   i.e. a critical immersion must be minimal.

2. We take now normal variations $\delta X = \psi\nu$ to get along the boundary $\partial\Sigma$,

$$0 = \oint_{\partial\Sigma} \left(J' \cdot [\psi\nu] - 2\varpi\tau_g' \partial_n \psi \right) ds\,,$$

   where $J := 2\alpha T'' + \left(3\alpha\kappa^2 + \varpi\tau_g^2 - \beta\right) T + 2\varpi\tau_g T \times T'$.
   This implies:

   - $J' \cdot \nu \equiv 0$ on $\partial\Sigma$, and
   - τ_g is (locally) constant. (Also if the frame is critical!)

Euler-Lagrange Equations (1)

1. We first consider compactly supported variations, obtaining

$$H \equiv 0 \,, \qquad \text{on } \Sigma \,,$$

   i.e. a critical immersion must be minimal.

2. We take now normal variations $\delta X = \psi\nu$ to get along the boundary $\partial\Sigma$,

$$0 = \oint_{\partial\Sigma} \left(J' \cdot [\psi\nu] - 2\varpi\tau_g' \partial_n \psi \right) ds \,,$$

   where $J := 2\alpha T'' + \left(3\alpha\kappa^2 + \varpi\tau_g^2 - \beta\right) T + 2\varpi\tau_g T \times T'$.
   This implies:

   - $J' \cdot \nu \equiv 0$ on $\partial\Sigma$, and
   - τ_g is (locally) constant. (Also if the frame is critical!)

3. Finally, from tangent variations we also get

$$J' \cdot n + \sigma \equiv 0 \,, \qquad \text{on } \partial\Sigma \,.$$

Euler-Lagrange Equations (2)

The Euler-Lagrange equations for equilibria of $E[X]$ are:

$$\begin{aligned} H &= 0\,, && \text{on } \Sigma\,, \\ J' \cdot \nu &= 0\,, && \text{on } \partial\Sigma\,, \\ J' \cdot n + \sigma &= 0\,, && \text{on } \partial\Sigma\,, \end{aligned}$$

together with τ_g being constant along the boundary.

Euler-Lagrange Equations (2)

The Euler-Lagrange equations for equilibria of $E[X]$ are:

$$\begin{aligned} H &= 0\,, && \text{on } \Sigma\,, \\ J' \cdot \nu &= 0\,, && \text{on } \partial\Sigma\,, \\ J' \cdot n + \sigma &= 0\,, && \text{on } \partial\Sigma\,, \end{aligned}$$

together with τ_g being constant along the boundary. It may have different (constant) values on different boundary components.

Euler-Lagrange Equations (2)

The Euler-Lagrange equations for equilibria of $E[X]$ are:

$$\begin{aligned} H &= 0\,, && \text{on } \Sigma\,, \\ J' \cdot \nu &= 0\,, && \text{on } \partial\Sigma\,, \\ J' \cdot n + \sigma &= 0\,, && \text{on } \partial\Sigma\,, \end{aligned}$$

together with τ_g being constant along the boundary. It may have different (constant) values on different boundary components.

- We can combine the last two Euler-Lagrange equations in a vector form, obtaining

$$J' + \sigma n = 0$$

  along the boundary $\partial\Sigma$.

Euler-Lagrange Equations (2)

The Euler-Lagrange equations for equilibria of $E[X]$ are:

$$\begin{aligned} H &= 0\,, && \text{on } \Sigma\,, \\ J' \cdot \nu &= 0\,, && \text{on } \partial\Sigma\,, \\ J' \cdot n + \sigma &= 0\,, && \text{on } \partial\Sigma\,, \end{aligned}$$

together with τ_g being constant along the boundary. It may have different (constant) values on different boundary components.

- We can combine the last two Euler-Lagrange equations in a vector form, obtaining

  $$J' + \sigma n = 0$$

  along the boundary $\partial\Sigma$. So, for each boundary component C,

  $$\oint_C n\, ds = 0$$

  holds.

Local Equilibrium Configurations

Local Equilibrium Configurations

Theorem (Palmer & —, 2020)

There exists a curve C and a minimal surface with boundary Σ, such that $C \subset \partial\Sigma$ and the Euler-Lagrange equations are satisfied along C.

Local Equilibrium Configurations

Theorem (Palmer & —, 2020)

There exists a curve C and a minimal surface with boundary Σ, such that $C \subset \partial\Sigma$ and the Euler-Lagrange equations are satisfied along C.

Idea of the proof (constructing the surface)

Local Equilibrium Configurations

Theorem (Palmer & —, 2020)

There exists a curve C and a minimal surface with boundary Σ, such that $C \subset \partial\Sigma$ and the Euler-Lagrange equations are satisfied along C.

Idea of the proof (constructing the surface):

1. The boundary condition $J' + \sigma n = 0$ can be written as a system (in κ_g and κ_n) of ODE with real analytic coefficients.

Local Equilibrium Configurations

Theorem (Palmer & —, 2020)

There exists a curve C and a minimal surface with boundary Σ, such that $C \subset \partial\Sigma$ and the Euler-Lagrange equations are satisfied along C.

Idea of the proof (constructing the surface):

1. The boundary condition $J' + \sigma n = 0$ can be written as a system (in κ_g and κ_n) of ODE with real analytic coefficients. Therefore, the solutions κ_g and κ_n are real analytic functions.

Local Equilibrium Configurations

Theorem (Palmer & —, 2020)

There exists a curve C and a minimal surface with boundary Σ, such that $C \subset \partial\Sigma$ and the Euler-Lagrange equations are satisfied along C.

Idea of the proof (constructing the surface):

1. The boundary condition $J' + \sigma n = 0$ can be written as a system (in κ_g and κ_n) of ODE with real analytic coefficients. Therefore, the solutions κ_g and κ_n are real analytic functions.
2. These functions together with the constant τ_g determine a (unique) real analytic curve $C(s)$ and a normal ν along C.

Local Equilibrium Configurations

Theorem (Palmer & —, 2020)

There exists a curve C and a minimal surface with boundary Σ, such that $C \subset \partial\Sigma$ and the Euler-Lagrange equations are satisfied along C.

Idea of the proof (constructing the surface):

1. The boundary condition $J' + \sigma n = 0$ can be written as a system (in κ_g and κ_n) of ODE with real analytic coefficients. Therefore, the solutions κ_g and κ_n are real analytic functions.
2. These functions together with the constant τ_g determine a (unique) real analytic curve $C(s)$ and a normal ν along C.
3. We now apply Björling's Formula

$$X(z) := \Re\left(C(z) + i\int_{s_o}^{z} C'(\omega) \times \nu(\omega)\, d\omega\right).$$

Global Results for Topological Discs

Let $X : \Sigma \cong D \to \mathbb{R}^3$ be an immersion of a topological disc D.

Global Results for Topological Discs

Let $X : \Sigma \cong D \to \mathbb{R}^3$ be an immersion of a topological disc D.

Theorem (Palmer & —, 2020)

Let $X : \Sigma \cong D \to \mathbb{R}^3$ be an immersion of a minimal surface of disc type having constant geodesic torsion along the boundary $\partial\Sigma$. Then, the surface is a planar domain.

Global Results for Topological Discs

Let $X : \Sigma \cong D \to \mathbb{R}^3$ be an immersion of a topological disc D.

Theorem (Palmer & —, 2020)

Let $X : \Sigma \cong D \to \mathbb{R}^3$ be an immersion of a minimal surface of disc type having constant geodesic torsion along the boundary $\partial\Sigma$. Then, the surface is a planar domain.

Corollary (Palmer & —, 2020)

Let $X : \Sigma \cong D \to \mathbb{R}^3$ be an immersion of a disc type critical for $E[X]$, then the surface is a compact domain in the plane

Global Results for Topological Discs

Let $X : \Sigma \cong D \to \mathbb{R}^3$ be an immersion of a topological disc D.

Theorem (Palmer & —, 2020)

Let $X : \Sigma \cong D \to \mathbb{R}^3$ be an immersion of a minimal surface of disc type having constant geodesic torsion along the boundary $\partial\Sigma$. Then, the surface is a planar domain.

Corollary (Palmer & —, 2020)

Let $X : \Sigma \cong D \to \mathbb{R}^3$ be an immersion of a disc type critical for $E[X]$, then the surface is a compact domain in the plane bounded by an area-constrained elastic curve.

Area-Constrained Elastic Curves

Area-Constrained Elastic Curves

- Associated Euler-Lagrange equation:

$$2\alpha\kappa'' + \left(\alpha\kappa^2 - \beta\right)\kappa \pm \sigma = 0\,.$$

Area-Constrained Elastic Curves

- Associated Euler-Lagrange equation:

$$2\alpha\kappa'' + \left(\alpha\kappa^2 - \beta\right)\kappa \pm \sigma = 0\,.$$

- Physically: elastic rods under a constant perpendicular force directed along their length (not only at the ends).

Area-Constrained Elastic Curves

- Associated Euler-Lagrange equation:

$$2\alpha\kappa'' + \left(\alpha\kappa^2 - \beta\right)\kappa \pm \sigma = 0\,.$$

- Physically: elastic rods under a constant perpendicular force directed along their length (not only at the ends).
- Alternative names: elasticae under pressure, buckled rings.

Area-Constrained Elastic Curves

- Associated Euler-Lagrange equation:

$$2\alpha\kappa'' + \left(\alpha\kappa^2 - \beta\right)\kappa \pm \sigma = 0\,.$$

- Physically: elastic rods under a constant perpendicular force directed along their length (not only at the ends).
- Alternative names: elasticae under pressure, buckled rings.

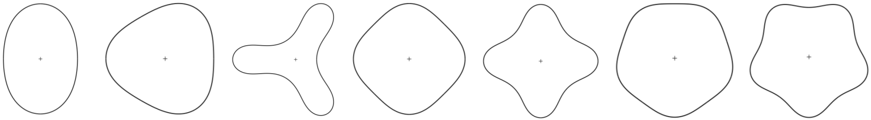
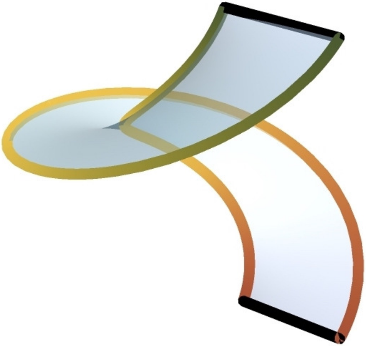


FIGURE: **(Wegner, 2019)**

Non Planar Example

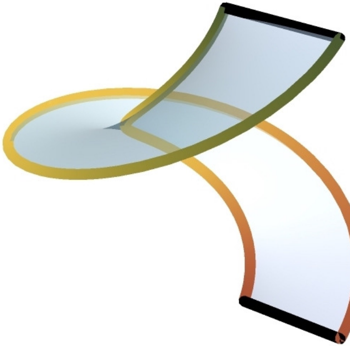
Non Planar Example

Consider domains in a minimal helicoid of the type:



Non Planar Example

Consider domains in a minimal helicoid of the type:



- They are critical for $E[X]$ having partially elastic boundary.

Intrinsic Variational Problem

Intrinsic Variational Problem

1. Fix a minimal immersion $X : \Sigma \to \mathbb{R}^3$.

Intrinsic Variational Problem

1. Fix a minimal immersion $X : \Sigma \to \mathbb{R}^3$.
2. Search for curves lying in the surface

Intrinsic Variational Problem

1. Fix a minimal immersion $X : \Sigma \to \mathbb{R}^3$.
2. Search for curves lying in the surface enclosing a compact domain of prescribed area Δ,

Intrinsic Variational Problem

1. Fix a minimal immersion $X : \Sigma \to \mathbb{R}^3$.
2. Search for curves lying in the surface enclosing a compact domain of prescribed area Δ, i.e. critical curves for

$$F[C] := \sigma \mathcal{A}[\Delta] + \oint_C \left(\alpha\kappa^2 + \varpi\tau_g^2 + \beta \right) ds\,.$$

Intrinsic Variational Problem

1. Fix a minimal immersion $X : \Sigma \to \mathbb{R}^3$.
2. Search for curves lying in the surface enclosing a compact domain of prescribed area Δ, i.e. critical curves for

$$F[C] := \sigma \mathcal{A}[\Delta] + \oint_C \left(\alpha\kappa^2 + \varpi\tau_g^2 + \beta\right) ds\,.$$

3. Assume $\alpha = \varpi$,

Intrinsic Variational Problem

1. Fix a minimal immersion $X : \Sigma \to \mathbb{R}^3$.
2. Search for curves lying in the surface enclosing a compact domain of prescribed area Δ, i.e. critical curves for

$$F[C] := \sigma \mathcal{A}[\Delta] + \oint_C \left(\alpha \kappa^2 + \varpi \tau_g^2 + \beta \right) ds \, .$$

3. Assume $\alpha = \varpi$, i.e. $\epsilon = 0$ (for instance, cork).

Intrinsic Variational Problem

1. Fix a minimal immersion $X : \Sigma \to \mathbb{R}^3$.
2. Search for curves lying in the surface enclosing a compact domain of prescribed area Δ, i.e. critical curves for

$$F[C] := \sigma \mathcal{A}[\Delta] + \oint_C \left(\alpha \kappa^2 + \varpi \tau_g^2 + \beta\right) ds \,.$$

3. Assume $\alpha = \varpi$, i.e. $\epsilon = 0$ (for instance, cork).
4. We have the intrinsic Kirchhoff-Plateau problem

$$\widetilde{F}[C] := \sigma \mathcal{A}[\Delta] + \oint_C \left(\alpha \left[\kappa_g^2 - K\right] + \beta\right) ds \,,$$

Intrinsic Variational Problem

1. Fix a minimal immersion $X : \Sigma \to \mathbb{R}^3$.
2. Search for curves lying in the surface enclosing a compact domain of prescribed area Δ, i.e. critical curves for

$$F[C] := \sigma \mathcal{A}[\Delta] + \oint_C \left(\alpha \kappa^2 + \varpi \tau_g^2 + \beta \right) ds \,.$$

3. Assume $\alpha = \varpi$, i.e. $\epsilon = 0$ (for instance, cork).
4. We have the intrinsic Kirchhoff-Plateau problem

$$\widetilde{F}[C] := \sigma \mathcal{A}[\Delta] + \oint_C \left(\alpha \left[\kappa_g^2 - K \right] + \beta \right) ds \,,$$

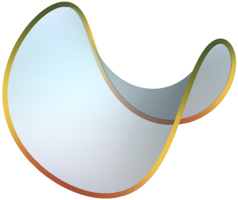
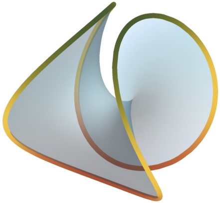
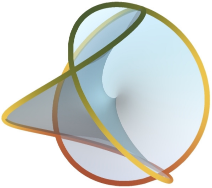
since the Gaussian curvature K along $\partial\Sigma$ is given by $K := -\kappa_n^2 - \tau_g^2$.

Enneper's Minimal Surface

For the fixed Enneper's minimal surface,

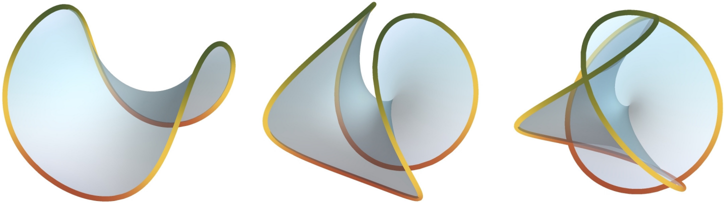
Enneper's Minimal Surface

For the fixed Enneper's minimal surface, we consider domains corresponding with planar circular discs:

Enneper's Minimal Surface

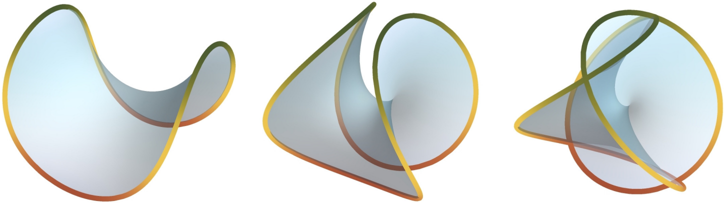
For the fixed Enneper's minimal surface, we consider domains corresponding with planar circular discs:



- They are critical for $\widetilde{F}[C]$. Here, the parameter β, i.e. the length of the boundary, is changing.

Enneper's Minimal Surface

For the fixed Enneper's minimal surface, we consider domains corresponding with planar circular discs:



- They are critical for $\widetilde{F}[C]$. Here, the parameter β, i.e. the length of the boundary, is changing.
- There are more examples. For instance: the catenoid.

THE END

- B. Palmer and A. Pámpano, Minimal Surfaces with Elastic and Partially Elastic Boundary, *Proc. A Royal Soc. Edinburgh*, DOI: https://doi.org/10.1017/prm.2020.56.

THE END

- B. Palmer and A. Pámpano, Minimal Surfaces with Elastic and Partially Elastic Boundary, *Proc. A Royal Soc. Edinburgh*, DOI: https://doi.org/10.1017/prm.2020.56.

Thank You!