The Euler-Plateau Problem with Elastic Modulus

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Minimal Surfaces

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where H is the mean curvature and ν is the unit normal to Σ. So, the immersion is minimal if and only if $H \equiv 0$.

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- The Euler-Plateau Problem. The boundary components of $\partial\Sigma$ are elastic: **(Giomi & Mahadevan, 2012)**

$$\mathcal{EP}[X] := \sigma\mathcal{A}[X] + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \beta\right) ds\,.$$

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The Euler-Plateau Problem with Elastic Modulus

For an immersion $X : \Sigma \to \mathbb{R}^3$ we consider the total energy

$$E[X] := \sigma \mathcal{A}[X] + \eta \int_\Sigma K \, d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \beta\right) ds \,,$$

where $\sigma > 0$, $\eta \neq 0$, $\alpha \geq 0$ and $\beta \in \mathbb{R}$.

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where τ_g denotes the geodesic torsion along the boundary.

- The vector field J is defined as **(Langer & Singer, 1984)**:

$$J := 2\alpha T'' + \left(3\alpha\kappa^2 - \beta\right) T\,.$$

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- Among them, there are always area-constrained elastic circles.

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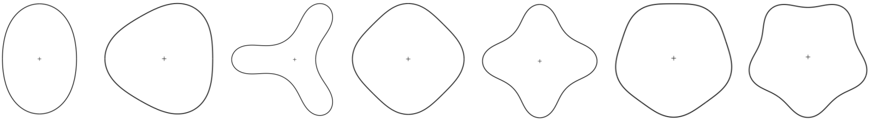

FIGURE: **(Wegner, 2019)**

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Using an argument involving the Cauchy-Kovalevskaya Theorem we can prove:

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- The condition $\beta < 0$ can also be obtained after rescaling.
- From now on, we will assume $\alpha > 0$ holds.

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Corollary

Axially symmetric critical surfaces are planar disks bounded by an area-constrained elastic circle.

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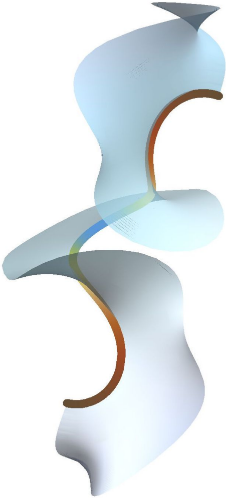

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Proposition

If the surface is a topological annulus, then $\tau_g > 0$ (or $\tau_g < 0$) holds everywhere.

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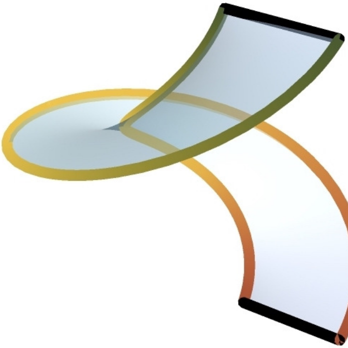

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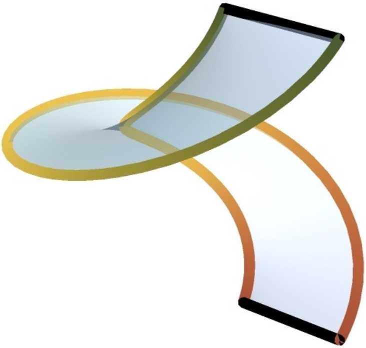

- They are critical for $E[X]$ having partially elastic boundary.

THE END

- A. Gruber, A. Pámpano and M. Toda, Regarding the Euler-Plateau Problem with Elastic Modulus, *Submitted*, ArXiv: arXiv: 2010.00149 [math.DG]

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Thank You!