On Some Open Problems Related to p-Elastic Curves

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- **1744**: L. Euler described the shape of planar elastic curves (partially solved by Jacob Bernoulli 1692–1694).

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- Case $p > 2$. (Applications: Willmore-Chen submanifolds, string theories,...)

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- Case $p = 1/3$. Equi-affine length and parabolas. (Blaschke, 1923).
- Cases $p = (n-2)/(n+1)$. Arise in the theory of biconservative hypersurfaces. (Montaldo & P., 2020; Montaldo, Oniciuc & P., 2022).

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The study of free p-elastic curves is a central topic in Differential Geometry and Calculus of Variations.

Variational Problem

Let $p \in \mathbb{R}$ and consider the functionals

$$\mathbf{\Theta}_p(\gamma) := \int_\gamma \kappa^p \, ds \, ,$$

acting on the space of non-null smooth immersed curves in $M_r^2(\rho)$.
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The Euler-Lagrange Equation

A critical point γ of $\mathbf{\Theta}_p$ must satisfy

$$p \frac{d^2}{ds^2}\left(\kappa^{p-1}\right) + \varepsilon_1 \varepsilon_2 (p-1)\kappa^{p+1} + \varepsilon_1 p \rho \kappa^{p-1} = 0 \,.$$

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First Integral

If γ is a critical point of $\boldsymbol{\Theta}_p$ with non-constant curvature κ then

$$p^2(p-1)^2\kappa^{2(p-2)}\left(\kappa'\right)^2 + \varepsilon_1\varepsilon_2(p-1)^2\kappa^{2p} + \varepsilon_1\rho p^2\kappa^{2(p-1)} = a\,,$$

must hold, where $a \in \mathbb{R}$. (The case $p = 2$ is special.)

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- In $\mathbb{H}^2$, there exist (non-trivial) closed free p-elastic curves if and only if $p > 1$. (P., Samarakkody & Tran, Preprint and Arroyo, Barros, Garay, Langer, Mencía, Musso, Singer,...).

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- In $\mathbb{S}^2_1$, there exist (non-trivial) closed p-elastic curves if and only if $p < 0$. (P., Samarakkody & Tran, Preprint).

Closure Condition

Theorem

For every pair of relatively prime natural numbers (n, m) satisfying $m < 2n < \sqrt{2}\, m$, there exists a non-trivial closed free p-elastic curve immersed in:

1. If $p > 1$, the hyperbolic plane $\mathbb{H}^2$.
2. If $p \in (0, 1)$, the round sphere $\mathbb{S}^2$.
3. If $p < 0$, the de Sitter 2-space $\mathbb{S}^2_1$.

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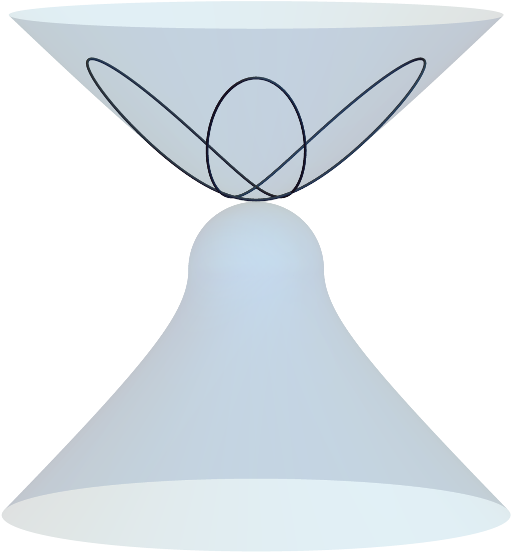
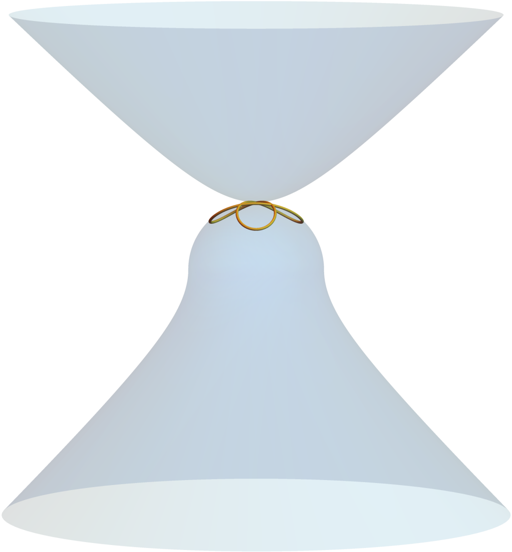
$$\Lambda_p(a) := 2p(p-1)^2\sqrt{|a|}\int_\beta^\alpha \frac{\kappa^{2(p-1)}}{\left(a - \varepsilon_1 \rho\, p^2 \kappa^{2(p-1)}\right)\sqrt{Q_{p,a}(\kappa)}}\, d\kappa\,.$$

Example $p = 2$

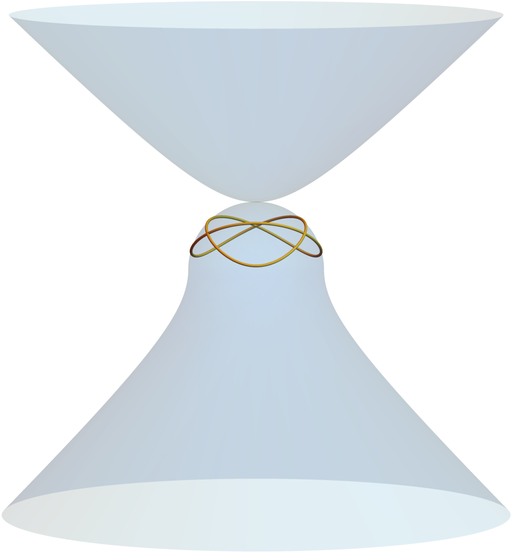


Example $p = 1.1$

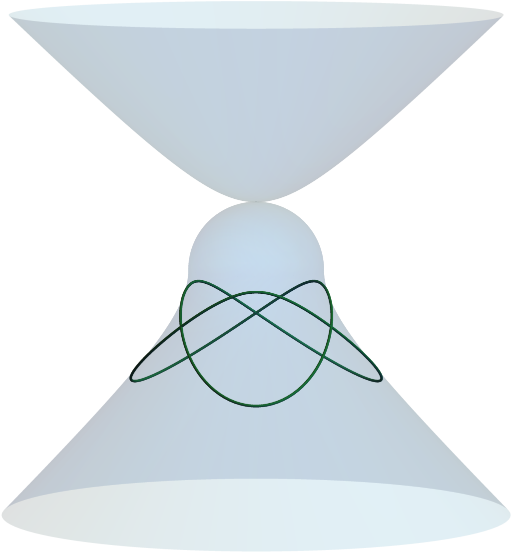
Example $p = 0.8$



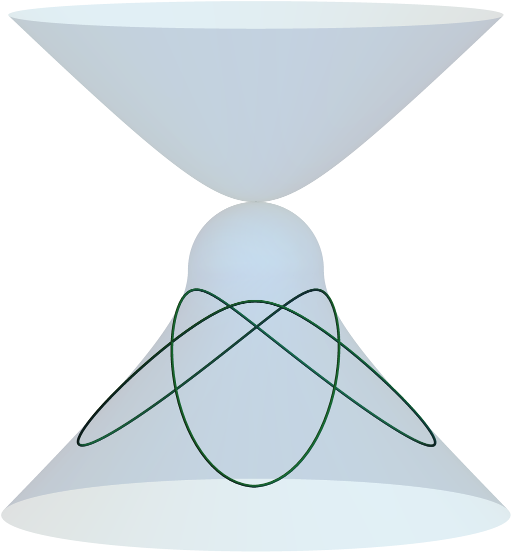
Example $p = 0.2$



Example $p = -0.5$



Example $p = -1$



THE END

Thank You!